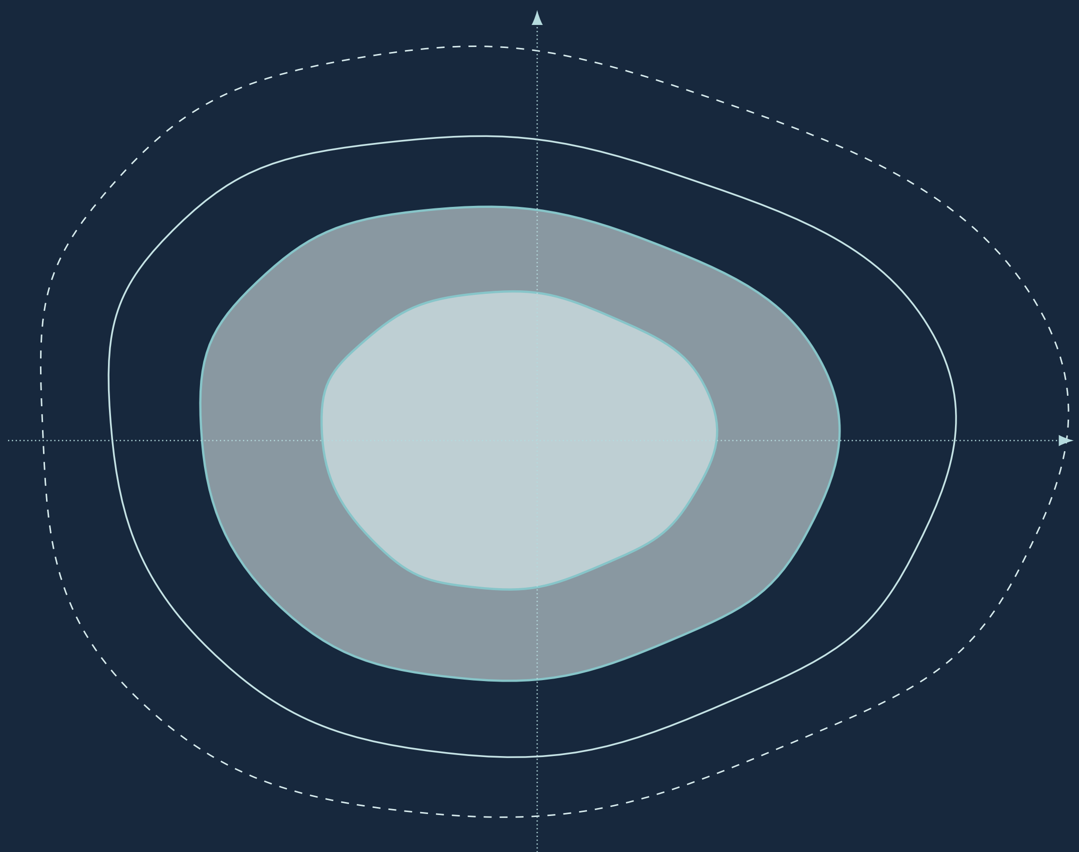
NESTED VIOLATION SETS

for the Generalized Damascus Inequality

An elementary proof of Conjecture 5.5



$$\mathcal{I}_n^3 \subseteq \mathcal{I}_{n+1}^3$$

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Abstract

For a positive integer n, define

$$\phi_n(t) = \frac{t^n - 1}{t^{n+1} + 1}, \qquad S_n(x, y, z) = \phi_n(x) + \phi_n(y) + \phi_n(z)$$

on the surface $xyz = 1$, $x, y, z > 0$. Sujsuntinukul and Chesneau conjectured that the violation sets

$$\mathcal{I}_n^3 = \{(x, y, z) \in (0, \infty)^3 : xyz = 1,\ S_n(x, y, z) > 0\}$$

are nested: $\mathcal{I}_n^3 \subseteq \mathcal{I}_{n+1}^3$. We prove the conjecture by an elementary multiplier argument. In fact, whenever $S_n(x, y, z) > 0$, one has the stronger strict inequality $S_{n+1}(x, y, z) > S_n(x, y, z) > 0$. The proof first identifies the only possible sign pattern of a violating triple and then compares the positive and negative summands through a reciprocal-invariant one-step multiplier. We also derive a geometric localization of every violation and document exact symbolic, rational, and elimination-based checks that are independent of the proof.

Keywords. Damascus inequality; symmetric inequality; violation set; monotonicity; reciprocal symmetry; computer algebra.
2020 Mathematics Subject Classification. 26D15.

1 Introduction

1.1 Background

The Damascus inequality originated with a three-variable inequality proposed by Fozi M. Dannan under the multiplicative constraint $xyz = 1$. Dannan and Sitnik gave several proofs and related variants in 2016 [1]. A later proof by Kalinin and Pankratova used one-variable estimates together with geometric-arithmetic convexity [2]. Their paper also asked for which positive integers n the exponent family

$$S_n(x, y, z) = \sum_{t \in \{x,y,z\}} \frac{t^n - 1}{t^{n+1} + 1} \tag{1}$$

is nonpositive for every positive triple with product one.

Sujsuntinukul and Chesneau subsequently placed this question in the m-variable framework

$$S_n^m(x_1, \ldots, x_m) = \sum_{j=1}^{m} \frac{x_j^n - 1}{x_j^{n+1} + 1}, \qquad \mathcal{H}_m = \left\{ x \in (0, \infty)^m : \prod_{j=1}^{m} x_j = 1 \right\}.$$

They classified the pairs (m, n) for which $S_n^m \le 0$ on $\mathcal{H}_m$ and studied the geometry of the complementary region [3]. In the three-variable case, $S_n \le 0$ holds identically for $n = 1, 2, 3$, while for every $n \ge 4$ there are triples for which $S_n > 0$. Thus the natural object for $n \ge 4$ is not merely a single counterexample but the full violation set

$$\mathcal{I}_n^3 = \{(x, y, z) \in \mathcal{H}_3 : S_n(x, y, z) > 0\}. \tag{2}$$

The cited work proves two useful topological facts for each fixed $n \ge 4$: $\mathcal{I}_n^3$ is bounded, and it remains a positive distance from each of the planes $x = 1$, $y = 1$, and $z = 1$. Numerical plots further suggest that these sets grow with the exponent. This led to Conjecture 5.5 of [3], which asserts

$$\mathcal{I}_n^3 \subseteq \mathcal{I}_{n+1}^3 \qquad (n \ge 1).$$

The purpose of this note is to prove that assertion and expose the simple mechanism behind the nesting.

1.2 Main result and contributions

The main theorem gives more than set inclusion: once a fixed triple becomes a violation, its value of S_n increases strictly at every later exponent.

Theorem 1 (Nested violation sets)**.** *Let $n \geq 1$ be an integer and let $x, y, z > 0$ satisfy $xyz = 1$. If*

$$S_n(x, y, z) > 0,$$

then

$$S_{n+1}(x, y, z) > S_n(x, y, z) > 0.$$

Consequently,

$$\mathcal{I}_n^3 \subseteq \mathcal{I}_{n+1}^3 \qquad \text{for every } n \geq 1.$$

The argument is self-contained and has two structural ingredients.

(i) A violating triple has exactly two coordinates above 1 and one below 1. The opposite nontrivial sign pattern is ruled out by an auxiliary hyperbolic-tangent-type function.

(ii) The change from ϕ_n to ϕ_{n+1} is multiplication by a positive factor ρ_n. This factor is invariant under $t \mapsto 1/t$ and strictly decreases on $(1, \infty)$. On a violating triple, the negative summand therefore receives a smaller multiplier than either positive summand.

Besides proving Conjecture 5.5, the argument gives a quantitative lower bound for $S_{n+1} - S_n$ and a new localization condition $(x-1)(y-1) < 1$ after the two coordinates above 1 are labeled x and y. The accompanying computations audit the algebra but are not proof steps.

1.3 Organization

Section 2 describes the product-one surface and the sign chambers relevant to violations. Section 3 develops the reciprocal identity and excludes the forbidden chamber. The multiplier is motivated and analyzed in Section 4; the proof of the main theorem appears in Section 5. Section 6 records consequences and limitations. Appendix A documents the reproducible computational checks, while Appendix B documents the machine-checked Lean proof.

2 Geometry of the constraint

The logarithmic coordinates

$$u = \log x, \qquad v = \log y, \qquad w = \log z$$

turn the constraint $xyz = 1$ into the plane $u + v + w = 0$. Eliminating $w = -u - v$ identifies $\mathcal{H}_3$ with the (u, v)-plane. The three lines $u = 0$, $v = 0$, and $u + v = 0$ divide this plane into six open chambers, corresponding to the possible strict orderings of the coordinates relative to 1.

Because $\phi_n(t)$ has the sign of $t - 1$, the signs of the three summands in S_n agree with those chambers. The product constraint rules out the patterns in which all coordinates lie on the same side of 1. Lemma 3 will show that a positive sum is possible only in the three chambers having two positive logarithmic coordinates and one negative logarithmic coordinate.

The reciprocal map $t \mapsto 1/t$ becomes reflection of a logarithmic coordinate across zero. This elementary symmetry is responsible for both the sign-pattern reduction and the comparison of one-step multipliers below.

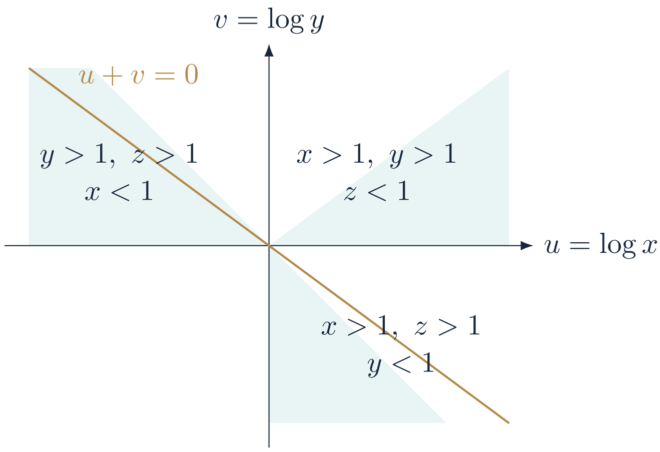

Figure 1: Sign chambers in logarithmic coordinates. The teal chambers are the only chambers that can contain violations; the shading is schematic and does not assert that every point in a shaded chamber violates the inequality.

3 Algebraic identities and sign reduction

Lemma 2 (Reciprocal identity)**.** *For every $t > 0$,*

$$\phi_n(1/t) = -t\phi_n(t).$$

Proof. Multiplying numerator and denominator by t^{n+1} gives

$$\phi_n(1/t) = \frac{t^{-n} - 1}{t^{-(n+1)} + 1} = \frac{t - t^{n+1}}{1 + t^{n+1}} = -t\frac{t^n - 1}{t^{n+1} + 1} = -t\phi_n(t).$$

□

Lemma 3 (The forbidden sign pattern)**.** *If $x, y, z > 0$, $xyz = 1$, and $S_n(x, y, z) > 0$, then, after permuting the coordinates,*

$$x > 1, \qquad y > 1, \qquad z < 1.$$

In particular, no coordinate equals 1, and the pattern with exactly one coordinate greater than 1 is impossible.

Proof. First suppose one coordinate equals 1. After relabeling, let $z = 1$, so $y = 1/x$. By Lemma 2,

$$S_n(x, 1/x, 1) = \phi_n(x) + \phi_n(1/x) = (1 - x)\phi_n(x).$$

Because $\operatorname{sgn}(\phi_n(x)) = \operatorname{sgn}(x - 1)$, this is strictly negative for $x \neq 1$ and equals zero for $x = 1$. Hence a triple with $S_n > 0$ has no coordinate equal to 1.

It remains to exclude the possibility that exactly one coordinate exceeds 1. Suppose, after relabeling,

$$x > 1, \qquad 0 < y < 1, \qquad 0 < z < 1.$$

Set

$$a = \frac{1}{y} > 1, \qquad b = \frac{1}{z} > 1.$$

The product condition gives $x = ab$. Define

$$\tau_n(t) = \frac{t^n - 1}{t^n + 1}.$$

For every $t > 1$, direct subtraction yields

$$\tau_n(t) - \phi_n(t) = \frac{t^n(t^n - 1)(t - 1)}{(t^n + 1)(t^{n+1} + 1)} > 0, \tag{3}$$

$$t\phi_n(t) - \tau_n(t) = \frac{(t^n - 1)(t - 1)}{(t^n + 1)(t^{n+1} + 1)} > 0. \tag{4}$$

Also, if $A = a^n > 1$ and $B = b^n > 1$, then

$$\begin{aligned} \tau_n(a) + \tau_n(b) - \tau_n(ab) &= \frac{A-1}{A+1} + \frac{B-1}{B+1} - \frac{AB-1}{AB+1} \\ &= \frac{(A-1)(B-1)(AB-1)}{(A+1)(B+1)(AB+1)} > 0. \end{aligned} \tag{5}$$

Using Lemma 2 and then (3)–(5),

$$\begin{aligned} S_n(ab, a^{-1}, b^{-1}) &= \phi_n(ab) - a\phi_n(a) - b\phi_n(b) \\ &< \tau_n(ab) - \tau_n(a) - \tau_n(b) \\ &< 0. \end{aligned}$$

Thus exactly one coordinate above 1 cannot occur when $S_n > 0$. Since $xyz = 1$, the only remaining possibility is exactly two coordinates above 1 and one below 1. □

4 The reciprocal multiplier

4.1 Why compare multiplicatively?

A direct attempt to compare S_{n+1} with S_n produces three rational differences with different signs. The sign reduction from Lemma 3 suggests a better organization. If, after relabeling, $x, y > 1 > z$, then two summands of S_n are positive and one is negative. What is needed is a common scale on which the exponent increment can be compared.

For a nonzero summand, define its relative one-step change by

$$\frac{\phi_{n+1}(t) - \phi_n(t)}{\phi_n(t)}.$$

The remarkable point is that this quotient extends to a positive function at $t = 1$, is unchanged by replacing t by $1/t$, and is ordered solely by the distance of t from 1 on the multiplicative scale. Thus it is precisely adapted to the product constraint. We denote this quotient by $\rho_n(t)$.

Lemma 4 (One-step multiplier)**.** *Define*

$$\rho_n(t) = \frac{t^n(t+1)}{(t^{n+2} + 1)(1 + t + \cdots + t^{n-1})} \qquad (t > 0).$$

Then $\rho_n(t) > 0$ *and*

$$\phi_{n+1}(t) - \phi_n(t) = \rho_n(t)\phi_n(t).$$

Moreover,

$$\rho_n(1/t) = \rho_n(t).$$

Proof. Direct subtraction gives

$$\begin{aligned} \phi_{n+1}(t) - \phi_n(t) &= \frac{t^{n+1} - 1}{t^{n+2} + 1} - \frac{t^n - 1}{t^{n+1} + 1} \\ &= \frac{t^n(t^2 - 1)}{(t^{n+2} + 1)(t^{n+1} + 1)}. \end{aligned}$$

Since

$$t^n - 1 = (t-1)(1 + t + \cdots + t^{n-1}),$$

dividing the last expression by $\phi_n(t)$ gives $\rho_n(t)$ for $t \neq 1$. At $t = 1$, both sides of the multiplier identity vanish, and the displayed formula gives $\rho_n(1) = 1/n$. Positivity is immediate.

For reciprocal invariance, substitute $1/t$ into the displayed formula for ρ_n and multiply numerator and denominator by the appropriate power of t; the result simplifies to $\rho_n(t)$. □

Lemma 5 (Monotonicity of the multiplier)**.** *The function ρ_n is strictly decreasing on $(1, \infty)$.*

Proof. For $t \neq 1$, exact simplification gives

$$1 + \frac{1}{\rho_n(t)} = \frac{t^{2n+2} - 1}{t^n(t^2 - 1)} = \sum_{k=0}^{n} t^{n-2k}. \tag{6}$$

Pair the terms with exponents m and $-m$. If $m > 0$, then for $t > 1$,

$$\frac{d}{dt}\left(t^m + t^{-m}\right) = m\left(t^{m-1} - t^{-m-1}\right) = mt^{-m-1}(t^{2m} - 1) > 0.$$

If n is even, the unpaired middle term is the constant 1. Therefore the Laurent sum on the right-hand side of (6) is strictly increasing on $(1, \infty)$. Hence $1/\rho_n(t)$ is strictly increasing there. Since $\rho_n(t) > 0$, ρ_n is strictly decreasing on $(1, \infty)$. □

Proposition 6 (Shape of the multiplier)**.** *For every $n \geq 1$, the function ρ_n has the following properties:*

$$0 < \rho_n(t) \leq \frac{1}{n} \qquad (t > 0),$$

with equality only at $t = 1$, and

$$\lim_{t \to 0^+} \rho_n(t) = \lim_{t \to \infty} \rho_n(t) = 0.$$

Equivalently, $\rho_n(e^u)$ is an even function of u that strictly decreases as $|u|$ increases away from zero.

Proof. Reciprocal invariance and Lemma 5 show that $t = 1$ is the unique maximum. The defining formula gives $\rho_n(1) = 1/n$. The limit at infinity is immediate from the degrees in that formula, and the limit at zero follows by reciprocal invariance. □

Remark 7 (The comparison mechanism). Suppose $x, y > 1$ and $z = (xy)^{-1}$. On the logarithmic line, the negative coordinate has magnitude

$$|\log z| = \log(xy) = \log x + \log y,$$

which is larger than either $\log x$ or $\log y$. Proposition 6 therefore says that the negative summand is multiplied by a strictly smaller relative increment than each positive summand. The proof of nesting is a weighted-sum version of this observation.

5 Proof of the conjecture

Proof of Theorem 1. Assume $S_n(x, y, z) > 0$. By Lemma 3, after permuting the coordinates we may suppose

$$x > 1, \qquad y > 1, \qquad z = (xy)^{-1} < 1.$$

Set

$$a = \phi_n(x) > 0, \qquad b = \phi_n(y) > 0, \qquad c = -\phi_n(z) > 0.$$

Then

$$S_n(x, y, z) = a + b - c > 0.$$

Write

$$r_x = \rho_n(x), \qquad r_y = \rho_n(y), \qquad r_z = \rho_n(z).$$

By reciprocal invariance and $z^{-1} = xy$,

$$r_z = \rho_n(z) = \rho_n(1/z) = \rho_n(xy).$$

Because $x, y > 1$, we have $xy > x$ and $xy > y$. Lemma 5 therefore gives

$$r_z < r_x, \qquad r_z < r_y.$$

Applying Lemma 4 to the three coordinates,

$$\begin{aligned} S_{n+1} - S_n &= r_x a + r_y b - r_z c \\ &= r_z(a + b - c) + (r_x - r_z)a + (r_y - r_z)b. \end{aligned}$$

Every term on the final line is positive. Consequently,

$$S_{n+1} - S_n > r_z S_n > 0.$$

Thus $S_{n+1} > S_n > 0$, proving the one-step inclusion. Iterating proves that a triple lying in $\mathcal{I}_n^3$ lies in $\mathcal{I}_{n'}^3$ for every $n' > n$. □

Corollary 8 (Quantitative strengthening)**.** *Under the hypotheses of Theorem 1, and with the coordinates labeled so that $x, y > 1 > z$,*

$$S_{n+1}(x, y, z) - S_n(x, y, z) > \rho_n(z) S_n(x, y, z) > 0.$$

Corollary 9 (Persistence and finite-step growth)**.** *If $(x, y, z) \in \mathcal{I}_n^3$, then for every integer $k \geq 1$,*

$$S_{n+k}(x, y, z) > S_n(x, y, z) \prod_{j=n}^{n+k-1} \big(1 + \rho_j(z)\big) > 0,$$

after the coordinates are labeled so that $x, y > 1 > z$.

Proof. Corollary 8 gives $S_{j+1} > (1 + \rho_j(z))S_j$ whenever $S_j > 0$. Theorem 1 preserves positivity, so iteration gives the displayed estimate. □

6 Consequences and geometric interpretation

6.1 An additional localization of violations

The multiplier proof has a useful consequence that is not visible from set inclusion alone. It bounds how far the two coordinates above 1 may move simultaneously.

Proposition 10 (Hyperbolic localization)**.** *Let $(x, y, z) \in \mathcal{I}_n^3$, with the coordinates labeled so that $x, y > 1 > z$. Then*

$$\frac{1}{x} + \frac{1}{y} > 1, \qquad \textit{equivalently} \qquad x + y > xy \qquad \textit{or} \qquad (x-1)(y-1) < 1. \tag{7}$$

Proof. For a fixed $t > 0$,

$$\lim_{j\to\infty} \phi_j(t) = \begin{cases} t^{-1}, & t > 1, \\ 0, & t = 1, \\ -1, & 0 < t < 1. \end{cases}$$

Theorem 1 gives $S_j(x, y, z) > S_n(x, y, z) > 0$ for every $j > n$. Taking $j \to \infty$ and using $x, y > 1 > z$ yields

$$\frac{1}{x} + \frac{1}{y} - 1 \geq S_n(x, y, z) > 0.$$

The three forms in (7) are algebraically equivalent. □

In logarithmic coordinates $x = e^u$, $y = e^v$ with $u, v > 0$, the last condition becomes

$$(e^u - 1)(e^v - 1) < 1.$$

Thus, even within a chamber permitted by Figure 1, violations are confined below a definite hyperbolic boundary. This complements the boundedness theorem in [3]: boundedness is proved there separately for each fixed exponent, whereas Proposition 10 supplies an exponent-independent necessary condition within each violating chamber. It is a localization, not a characterization; points satisfying (7) need not lie in any $\mathcal{I}_n^3$.

6.2 What the theorem does and does not say

The theorem orders the values of S_n only after they become positive. It does not assert that $S_{n+1}(x, y, z) > S_n(x, y, z)$ for every point of $\mathcal{H}_3$. Indeed, away from the violation region the three signed contributions need not be ordered in the way used in the proof. Nor does the theorem determine whether every inclusion $\mathcal{I}_n^3 \subseteq \mathcal{I}_{n+1}^3$ is proper.

The proof is especially three-dimensional. Once a violating triple is reduced to $x, y > 1 > z$, the product condition identifies the lone negative coordinate as $z^{-1} = xy$. Consequently its multiplier is compared with exactly two positive multipliers. In m variables, a sign chamber can contain several coordinates on each side of 1. Reciprocal invariance still holds term by term, but the product constraint no longer pairs the positive and negative parts in a way that immediately yields the same weighted comparison. The questions of boundedness and nesting for general m raised in [3] therefore require additional structure.

6.3 Conceptual summary of the proof

The full argument can be compressed into the following chain, with every arrow justified by a separate lemma:

$$\begin{aligned} S_n > 0 \Longrightarrow x, y > 1 > z \Longrightarrow z^{-1} = xy > \max\{x, y\} \\ \Longrightarrow \rho_n(z) < \min\{\rho_n(x), \rho_n(y)\} \Longrightarrow S_{n+1} > S_n. \end{aligned}$$

The first implication is a sign-pattern theorem; the middle implications use the product constraint, reciprocal symmetry, and monotonicity; the last is a positive weighted comparison. This separation of roles is also useful for formal verification, because the final comparison is elementary once the analytic properties of ρ_n are available.

7 Conclusion

Conjecture 5.5 of Sujsuntinukul and Chesneau follows from a reciprocal multiplier that becomes smaller as a coordinate moves farther from 1 on the multiplicative scale. A violating triple necessarily has two positive summands and one negative summand; the product-one constraint places the negative coordinate farther from 1 than either positive coordinate. Its relative one-step increase is therefore smaller, so positivity not only persists but strengthens strictly with the exponent.

The argument also gives a finite-step growth estimate and the necessary condition $(x-1)(y-1) < 1$ in each violating chamber. Natural next questions include whether any of the inclusions are strict in a uniform sense, whether the boundaries of $\mathcal{I}_n^3$ converge to the hyperbola in Proposition 10, and which parts of the multiplier method can survive when more than three variables are present.

A Independent computational checks

The proof above is elementary and does not depend on computation. The checks below were used during development to audit identities, compare independent implementations, and search for possible counterexamples. Reproducible scripts and recorded outputs accompany this paper. SymPy is used for exact symbolic calculation [6]; the optional nonlinear query uses Z3 [7].

A.1 Verification design

The checks are deliberately divided by logical role. An exact computer algebra identity can confirm a simplification, but checking finitely many values of n cannot prove a statement for every n. Exact rational sampling avoids floating-point error, but it remains sampling. Elimination and Sturm sequences give exact conclusions for the specific boundary system analyzed, not for all exponents. The following table records these distinctions.

Check	Arithmetic	Role in the paper
General identities	symbolic rational functions	Audits formulas used in the proof for representative n; the displayed algebra proves them generally.
Rational search	`Fraction` arithmetic	Independent counterexample search with $xyz = 1$ imposed exactly; evidence only.
$n = 4$ certificates	factorization and Bernstein basis	Exact local checks of multiplier positivity and monotonicity.
Boundary elimination	Gröbner bases, resultants, Sturm counts	Exact study of the special system $S_4 = S_5 = 0$.
NLSat query	exact nonlinear real arithmetic	Exploratory solver query; the recorded result is `unknown`, so no conclusion is taken from it.

Table 1: Scope and epistemic role of the computational checks.

A.2 Recorded checks

1. **Exact symbolic identities.** SymPy 1.14 verified the reciprocal identity, multiplier identity, reciprocal invariance of ρ_n, and Laurent identity for $n = 1, \ldots, 30$. The three rational factorizations used in Lemma 3 were also checked exactly. These tests guard against transcription errors; the symbolic proofs in the main text are uniform in n.

2. **Exact rational counterexample search.** A separate implementation using Python's `fractions.Fraction` evaluated rational triples from a fixed pseudorandom seed, with $xyz = 1$ enforced exactly. It tested 60,000 triples over $n = 1, \ldots, 30$; 10,276 of the evaluated pairs had $S_n > 0$, and none had $S_{n+1} \le 0$. This implementation uses only the Python standard library and is independent of SymPy. It is evidence only.

3. **Floating-point stress search.** A log-coordinate sampler tested 100,000 points for each $n = 4, \ldots, 20$ over the square $[-12, 12]^2$ in (u, v) coordinates, with $w = -u - v$. No sampled violation at level n became nonpositive at level $n+1$. The fixed seed makes the run reproducible, but the result has no deductive role.

4. **SOS certificate for $n = 4$.** The multiplier difference factors as

$$\rho_4(x) - \rho_4(xy) = \frac{x^4(y^2-1)(x^4y^2-1)Q(x,y)}{D(x,y)},$$

where $D(x, y) > 0$ for $x, y > 0$ and

$$Q(x,y) = x^4y^4 + x^4y^2 + x^2y^2 + y^2 + 1 = (x^2y^2)^2 + (x^2y)^2 + (xy)^2 + y^2 + 1^2.$$

This is an exact sum-of-squares certificate. The same factor appears as a local lemma in the Lean formalization described in Appendix B.

5. **Bernstein certificate.** After the compactification $t = (1+s)/(1-s)$, the cleared numerator of $-\rho_4'(t)$ is

$$8s(s+1)^3(5s^4 + 22s^2 + 5),$$

whose degree-eight Bernstein coefficients on $[0, 1]$ are

$$0,\ 5,\ \frac{100}{7},\ \frac{232}{7},\ \frac{2624}{35},\ \frac{1200}{7},\ \frac{2752}{7},\ 896,\ 2048.$$

All nonconstant coefficients are positive.

6. **Gröbner bases, resultants, Sturm sequences, and exact intervals.** For the boundary system $S_4 = S_5 = 0$, elimination in elementary symmetric coordinates produced an eliminant $(p-3)(p-1)H_{22}(p)$. Its unique algebraic root with $p > 3$ induces a negative cubic discriminant, certified by rational interval arithmetic, and therefore does not represent a real positive triple. Direct elimination in x, y produced resultant factors of degrees $1, 1, 2, 4, 4, 66$; exact Sturm counts found no nontrivial projection root with $x > 1$. For background on these algebraic tools, see [4, 5].

7. **CAD/NLSat.** An exact quantifier-free nonlinear real arithmetic query for the $n = 4$ counterexample region was submitted to Z3's NLSat backend. It returned `unknown` under the imposed ten-second timeout. The Wolfram Language file in the repository records a corresponding CAD query, but it was not executed in the present environment. No CAD result is claimed. The general theorem is settled by the elementary proof above.

A.3 Reproduction

From the repository root, the standard audit is run with

```
python verification/symbolic_audit.py
python verification/fraction_audit.py
```

The longer elimination, randomized, and NLSat checks are enabled by

```
python verification/symbolic_audit.py --heavy \
  --random 100000 --nlsat 10000
```

The recorded output also states the Python and package versions. Because computer algebra output can depend on implementation details, the repository pins the principal dependencies and runs the standard checks in continuous integration.

B Machine-checked formalization

The repository contains a complete Lean 4 formalization, using the proof assistant described by de Moura and Ullrich [8]. The file defines ϕ_n, S_n, the auxiliary function τ_n, the paired Laurent polynomial, and the multiplier ρ_n. It proves the reciprocal identity, the forbidden sign pattern, reciprocal invariance and strict monotonicity of the multiplier, the one-step multiplier identity, and the final weighted comparison. The declaration `nested_violation_sets` is a direct formal counterpart of Theorem 1: for positive x, y, z with $xyz = 1$ and positive integer n, it proves

$$S_n(x, y, z) > 0 \quad \Longrightarrow \quad S_{n+1}(x, y, z) > S_n(x, y, z).$$

The formalization also retains the five-square $n = 4$ certificate and the positivity result for its cleared multiplier numerator. Lean and Mathlib are pinned to version 4.32.1, and the checked build command is `lake build --wfail`. The source contains no `sorry` placeholders or custom axioms. Continuous integration is configured to rebuild the proof with warnings treated as errors and to validate the compiled environment with Lean's standalone `leanchecker`. The symbolic and exact-rational audits in Appendix A remain separate checks rather than assumptions of the formal proof.

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