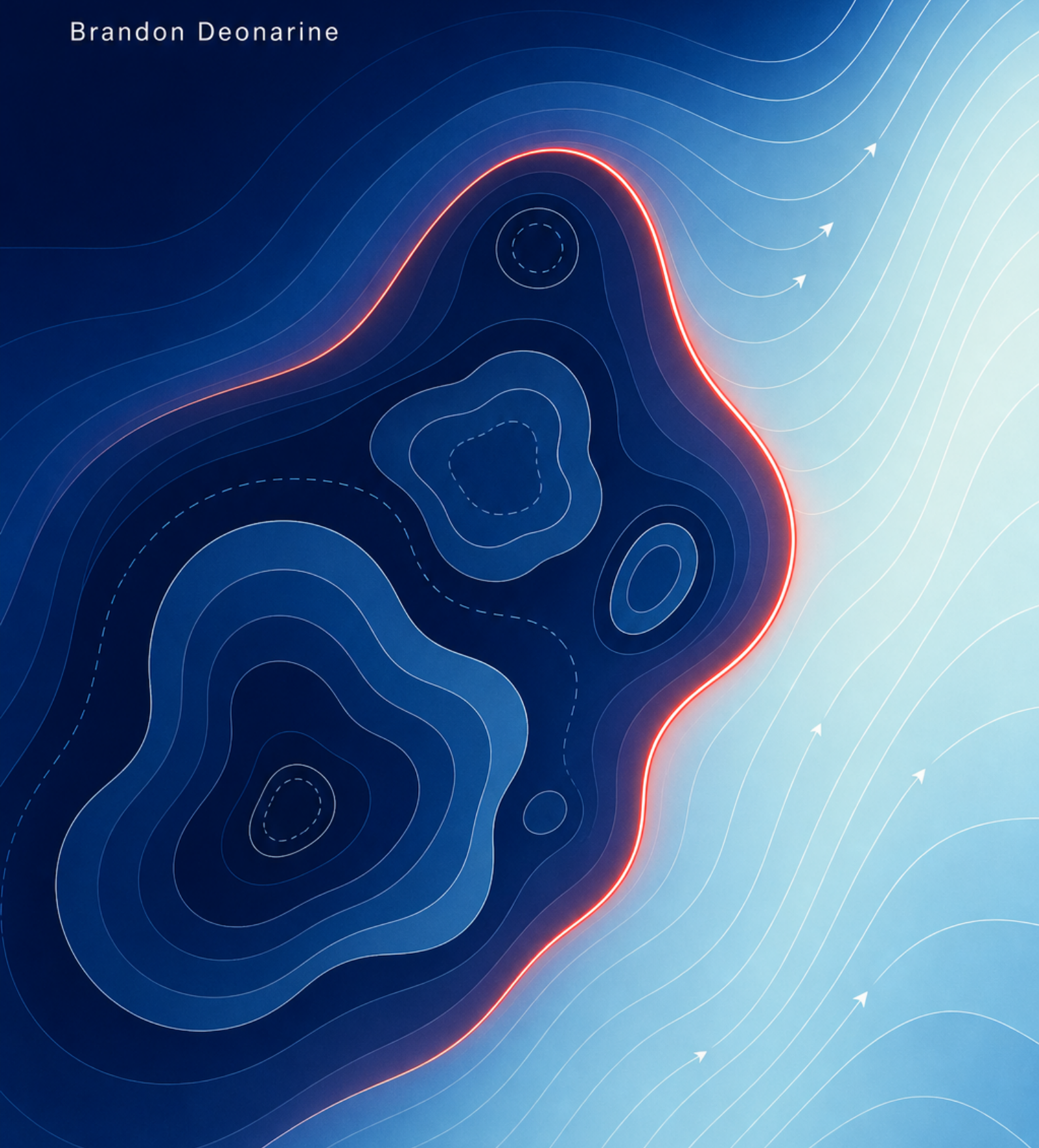


Geometry, Escape, and Phase Transitions in Violation Sets for the Generalized Damascus Inequality

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Abstract

For integers $m, n \geq 1$, let

$$S_n^m(x) = \sum_{j=1}^m \frac{x_j^n - 1}{x_j^{n+1} + 1}, \quad \mathcal{H}_m = \{x_j > 0 : \prod_j x_j = 1\},$$

and write $\mathcal{I}_n^m = \{S_n^m > 0\}$. Building on the validity classification of Sujsuntinukul and Chesneau and the three-variable nestedness theorem, we give a structural classification of higher-dimensional violation sets. The exact global escape energy is $-1 + (m-2)M_n$, where $M_n = \max_{t>0} (t^n - 1)/(t^{n+1} + 1)$; a sign sector with r coordinates above 1 has escape energy $-1 + (r-1)M_n$. These formulas yield complete boundedness and separation phase diagrams and a detailed second-order analysis of the unique critical lattice point $(m, n) = (5, 2)$.

Exponent dynamics undergo a sharp dimensional transition. Universal nestedness holds exactly for $m \leq 3$, while in four variables $\mathcal{I}_n^4 \subseteq \mathcal{I}_{n+1}^4$ exactly for $n \leq 6$. More generally, if d_n is the least dimension in which one-step nesting fails, then

$$d_1 = 12, \quad d_2 = 7, \quad d_3 = 6, \quad d_4 = d_5 = d_6 = 5, \quad d_n = 4 \quad (n \geq 7).$$

The finite lower-cell inclusions are computer-assisted theorems established by deterministic exact rational Bernstein subdivision. Their verification package is specified in Section 12: the four-variable and topology arguments have archived replayable proof trees, while the early staircase is checked by a deterministic exact exhaustive program whose full source, pinned dependencies, manifest, and successful transcript are supplied.

A universal cubic blow-up gives sharp n^{-2} nucleation constants. For $m = 3$, the fixed set $\mathcal{I}_4^3$ has exactly three connected components, each contractible; the eventual violation envelope is an explicit triangle in reciprocal coordinates; the coordinate-plane gap is asymptotic to $4/n^2$; and the outer radius is asymptotic to $n/\log n$. Continuous exponents yield a real-parameter three-variable filtration and the exact first nonempty exponent $\nu_c = 3.9826231561\dots$, an exact four-variable failure $\mathcal{I}_{15/2}^4 \not\subseteq \mathcal{I}_8^4$, and the absence of an infinite forward four-variable inclusion ray from every $\nu > \nu_c$. At each fixed point re-entry complexity is finite, while a telescoping multiscale construction realizes N membership changes in dimension $2N + 4$ with at most $N + 3$ reciprocal radii. We also give an exact witness for an off-axis fixed-product bifurcation at exponent 7 in three variables.

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Keywords: Damascus inequality; symmetric inequalities; violation sets; Bernstein certificates; phase transitions; asymptotic geometry.

2020 Mathematics Subject Classification: 26D15.

1 Introduction

The Damascus inequality and its variants form a useful testing ground for the interaction of multiplicative constraints, one-variable shape, and symmetric inequalities. Dannan and Sitnik introduced and studied the original problem in 2016 [1]. A later simple proof was given by Kalinin and Pankratova [2]. Sujsuntinukul and Chesneau subsequently considered the family

$$S_n^m(x_1, \dots, x_m) = \sum_{j=1}^m \phi_n(x_j), \quad \phi_n(t) = \frac{t^n - 1}{t^{n+1} + 1}, \quad (1)$$

under the constraint

$$\mathcal{H}_m = \left\{ (x_1, \dots, x_m) \in (0, \infty)^m : \prod_{j=1}^m x_j = 1 \right\}. \quad (2)$$

They classified exactly when $S_n^m \leq 0$ globally and introduced the open violation set

$$\mathcal{I}_n^m = \{x \in \mathcal{H}_m : S_n^m(x) > 0\}. \quad (3)$$

Throughout, “bounded” for a subset $A \subset \mathcal{H}_m$ means relatively bounded away from both 0 and ∞ : equivalently, there is $C > 1$ such that

$$C^{-1} \leq x_j \leq C \quad (x \in A, 1 \leq j \leq m).$$

This is the same as boundedness in logarithmic coordinates. We equip $\mathcal{H}_m$ with the logarithmic sup metric

$$d_{\log}(x, y) = \|\log x - \log y\|_{\infty}.$$

Whenever a Euclidean norm is needed it is written explicitly as $\|\cdot\|_2$; in particular, the critical stability sphere in Section 6 is Euclidean. We say that A is *separated from the coordinate planes* if

$$\inf_{x \in A} \min_j |\log x_j| > 0.$$

A strict sign sector $\Sigma_{r,s} \subset \mathcal{H}_m$ has exactly r coordinates above 1 and exactly s below 1; coordinates equal to 1 belong to lower-dimensional boundary strata and are treated separately through coordinate traces. Their classification is recalled in Table 1. In the nonempty three-variable cases they proved that $\mathcal{I}_n^3$ is separated from the coordinate planes $x_j = 1$ and is bounded. They also conjectured that the three-variable violation sets are nested in n and explicitly asked whether separation, boundedness, and increasing exponent evolution persist for arbitrary m [8].

An unpublished submitted companion manuscript by the present author, *Nested Violation Sets for the Generalized Damascus Inequality* (2026), proves the three-variable nestedness conjecture and its strict strengthening $S_{n+1}^3 > S_n^3 > 0$ on violations; a public copy is available at <https://brandondeonarine.com/papers/nested-violation-sets.pdf>. In accordance with the target journal’s reference policy, this unpublished manuscript is mentioned in the text rather than included in the reference list. The present paper is a sequel: it uses the reciprocal multiplier mechanism as a starting point, but its principal contributions concern higher-dimensional escape, separation, transient exponent dynamics, continuous exponents, and asymptotic geometry. Only the minimal multiplier facts needed for those extensions are restated here.

Table 1. Pairs for which $\mathcal{I}_n^m = \emptyset$, from Sujstuninukul–Chesneau [8].

m	valid exponents n
1	all $n \geq 1$
2	all $n \geq 1$
3	1, 2, 3
4	1
5	1
$m \geq 6$	none

The aim of this paper is not merely to answer those three questions independently, but to identify the geometry controlling all three. Three mechanisms recur throughout:

- (i) a reciprocal one-step multiplier that orders exponent increments by logarithmic distance from 1;
- (ii) an exact escape energy Λ_n^m controlling behaviour at infinity;
- (iii) a universal cubic blow-up Q_m controlling the first appearance of violations near the identity.

These mechanisms reveal a genuine phase transition at $m = 4$. Dimension three is the last dimension in which every possible violation is trapped in a persistent sign sector, and it is simultaneously the asymptotically critical confinement dimension.

Main results.. The principal conclusions are:

- exact global and sign-sector escape energies, including boundedness iff $(m - 2)M_n < 1$;
- exact separation among nonempty sets: precisely $m = 3, n \geq 4$, $(4, 2)$, $(4, 3)$, and $(6, 1)$;
- universal nestedness in all exponents iff $m \leq 3$, together with the sharp four-variable threshold $\mathcal{I}_n^4 \subseteq \mathcal{I}_{n+1}^4$ iff $n \leq 6$;
- the complete minimum-dimension failure staircase

$$d_n = 12, 7, 6, 5, 5, 5, 4, 4, \dots$$

for $n = 1, 2, 3, 4, 5, 6, 7, 8, \dots$, with the finite lower cells established by deterministic exact rational Bernstein subdivision;

- exact eventual exponent dynamics from a radial decomposition, a finite bound on membership changes in terms of the number of reciprocal radii, and a telescoping-pulse construction producing N changes in dimension $2N + 4$ with at most $N + 3$ radii;
- a complete classification of fixed transverse directions on the second-order critical sphere $\|y\|_2^2 = 72$ at the critical $(5, 2)$ escape profile;
- a fixed-exponent topology theorem: $\mathcal{I}_4^3$ has exactly three connected components, each contractible;
- a continuous-exponent interpolation in which one-below violations are strictly persistent, the three-variable sets form a continuous increasing filtration with exact onset $\nu_c = 3.9826231561\dots$, and every four-variable inclusion starting at $\nu > \nu_c$ fails for all sufficiently large target exponents;
- a sharp sign-sector nucleation law $6r/(r - 1)$, giving the global local constant $6(m - 1)/(m - 2)$, and the distinct coordinate-plane constant 4 for $m = 3$;
- $m = 3$ outer-radius growth $R_n \sim n/\log n$.

2 Algebraic preliminaries and multiplier geometry

Lemma 2.1 (Reciprocal identity). *For every $t > 0$,*

$$\phi_n(1/t) = -t\phi_n(t).$$

Proof. Multiplication of numerator and denominator by t^{n+1} gives

$$\phi_n(1/t) = \frac{t - t^{n+1}}{1 + t^{n+1}} = -t \frac{t^n - 1}{t^{n+1} + 1}.$$

□

Proposition 2.2 (Strict dissipation under coordinatewise inversion). *For $x = (x_1, \dots, x_m) \in \mathcal{H}_m$, write $x^{-1} = (x_1^{-1}, \dots, x_m^{-1})$. Then*

$$S_n^m(x) + S_n^m(x^{-1}) = - \sum_{j=1}^m \frac{(x_j - 1)^2(1 + x_j + \dots + x_j^{n-1})}{x_j^{n+1} + 1}. \quad (4)$$

Hence the right side is strictly negative away from the identity. In particular,

$$x \in \mathcal{I}_n^m \implies S_n^m(x^{-1}) < -S_n^m(x) < 0.$$

Proof. By Lemma 2.1,

$$\phi_n(t) + \phi_n(1/t) = (1-t)\phi_n(t) = - \frac{(t-1)^2(1+t+\dots+t^{n-1})}{t^{n+1}+1}.$$

Sum over the coordinates. □

Lemma 2.3 (One-step multiplier). *Define*

$$\rho_n(t) = \frac{t^n(t+1)}{(t^{n+2}+1)(1+t+\dots+t^{n-1})}.$$

Then $\rho_n(t) > 0$ and

$$\phi_{n+1}(t) - \phi_n(t) = \rho_n(t)\phi_n(t).$$

Moreover $\rho_n(1/t) = \rho_n(t)$, and ρ_n is strictly decreasing on $(1, \infty)$.

Proof. Direct subtraction gives

$$\phi_{n+1}(t) - \phi_n(t) = \frac{t^n(t^2 - 1)}{(t^{n+2} + 1)(t^{n+1} + 1)},$$

and division by $\phi_n(t)$ yields the stated multiplier. Reciprocal invariance follows by substitution.

For monotonicity, exact simplification yields

$$1 + \frac{1}{\rho_n(t)} = \frac{t^{2n+2} - 1}{t^n(t^2 - 1)} = \sum_{k=0}^n t^{n-2k}. \quad (5)$$

For $t > 1$, every paired term $t^r + t^{-r}$ with $r > 0$ is strictly increasing; if n is even the unpaired middle term is 1. Hence the right side of (5) is strictly increasing, so ρ_n is strictly decreasing. □

It is useful to interpret Lemma 2.3 radially. Put

$$r(t) = \min(t, 1/t) = e^{-|\log t|} \in (0, 1].$$

By reciprocal invariance and monotonicity, $\rho_n(t)$ is strictly increasing as $r(t)$ increases toward 1.

3 Sign-sector rigidity and persistence

Lemma 3.1 (Strict subadditivity of an auxiliary function). *For $t > 1$ define*

$$\tau_n(t) = \frac{t^n - 1}{t^n + 1}.$$

Then for $a, b > 1$,

$$\tau_n(ab) < \tau_n(a) + \tau_n(b).$$

Also

$$\phi_n(t) < \tau_n(t) < t\phi_n(t), \quad t > 1.$$

Proof. The two comparison inequalities follow from

$$\tau_n(t) - \phi_n(t) = \frac{t^n(t^n - 1)(t - 1)}{(t^n + 1)(t^{n+1} + 1)} > 0,$$

$$t\phi_n(t) - \tau_n(t) = \frac{(t^n - 1)(t - 1)}{(t^n + 1)(t^{n+1} + 1)} > 0.$$

Writing $A = a^n$, $B = b^n$ gives

$$\tau_n(a) + \tau_n(b) - \tau_n(ab) = \frac{(A - 1)(B - 1)(AB - 1)}{(A + 1)(B + 1)(AB + 1)} > 0.$$

□

Theorem 3.2 (Majorization obstruction). *Let $x \in \mathcal{H}_m$. Write the positive logarithms of the coordinates above 1 as*

$$u_1, \dots, u_r > 0,$$

and the absolute logarithms of the coordinates below 1 as

$$v_1, \dots, v_s > 0.$$

Pad the shorter vector by zeros so that the two vectors have the same length. The product constraint gives

$$\sum_i u_i = \sum_j v_j.$$

If u majorizes v , then $S_n^m(x) < 0$ for every $n \geq 1$. Consequently every violation must fail this majorization relation.

Proof. Set

$$F_n(w) = \tau_n(e^w) = \tanh \frac{nw}{2}, \quad w \geq 0.$$

The function F_n is strictly increasing, strictly concave on $(0, \infty)$, and $F_n(0) = 0$. By Karamata's inequality [4], $u \succ v$ implies

$$\sum_i F_n(u_i) \leq \sum_j F_n(v_j).$$

On the other hand, Lemma 3.1 and reciprocity give the strict sandwich

$$S_n^m(x) = \sum_i \phi_n(e^{u_i}) - \sum_j e^{v_j} \phi_n(e^{v_j}) < \sum_i F_n(u_i) - \sum_j F_n(v_j) \leq 0.$$

□

Corollary 3.3 (Spread inversion in the four-variable transient sector). *Suppose $x_1, x_2 > 1 > x_3, x_4$ and $x_1 x_2 x_3 x_4 = 1$. Write*

$$\log x_1 = u_1, \quad \log x_2 = u_2, \quad -\log x_3 = v_1, \quad -\log x_4 = v_2,$$

with each pair ordered decreasingly. If $S_n^4(x) > 0$, then

$$u_1 < v_1,$$

equivalently

$$|u_1 - u_2| < |v_1 - v_2|.$$

Thus the two coordinates above 1 must be strictly less spread in logarithmic distance than the reciprocal radii of the two coordinates below 1.

Proof. The two vectors have the same sum. For two-component vectors, u majorizes v exactly when $u_1 \geq v_1$. Apply Theorem 3.2; the spread formulation follows from equality of the pair sums. $\square$

Theorem 3.4 (One-above violations are impossible). *Let $x \in \mathcal{H}_m$. If exactly one coordinate exceeds 1 and every other coordinate is below 1, then*

$$S_n^m(x) < 0$$

for every $n \geq 1$.

Proof. After permutation let $x_1 > 1$ and put $a_j = 1/x_j > 1$ for $j \geq 2$. The constraint gives $x_1 = P := \prod_{j=2}^m a_j$. By Lemma 2.1,

$$S_n^m = \phi_n(P) - \sum_{j=2}^m a_j \phi_n(a_j).$$

Using Lemma 3.1 and iterating subadditivity,

$$S_n^m < \tau_n(P) - \sum_{j=2}^m \tau_n(a_j) \leq 0.$$

The first inequality is strict for every $m \geq 2$; the final inequality may be equality only in the single-factor case $m = 2$. Hence $S_n^m < 0$ in all cases. $\square$

Theorem 3.5 (Persistence in the one-below sector). *Suppose $x_1, \dots, x_{m-1} > 1$ and*

$$z = (x_1 \cdots x_{m-1})^{-1} < 1.$$

If $S_n^m(x_1, \dots, x_{m-1}, z) > 0$, then

$$S_{n+1}^m > S_n^m > 0.$$

More precisely,

$$S_{n+1}^m - S_n^m > \rho_n(z) S_n^m.$$

Proof. Set $a_i = \phi_n(x_i) > 0$ and $c = -\phi_n(z) > 0$. By reciprocal invariance,

$$\rho_n(z) = \rho_n(1/z) = \rho_n(x_1 \cdots x_{m-1}).$$

Since the product exceeds each x_i , Lemma 2.3 gives $\rho_n(z) < \rho_n(x_i)$. Hence

$$\begin{aligned} S_{n+1}^m - S_n^m &= \sum_i \rho_n(x_i) a_i - \rho_n(z) c \\ &= \rho_n(z) \left(\sum_i a_i - c \right) + \sum_i (\rho_n(x_i) - \rho_n(z)) a_i \\ &> \rho_n(z) S_n^m > 0. \end{aligned}$$

$\square$

Corollary 3.6 (Radial-order criterion). *Let $x \in \mathcal{I}_n^m$. If every coordinate above 1 is no farther from 1 in logarithmic distance than every coordinate below 1, then $S_{n+1}^m > S_n^m > 0$.*

Proof. Write $A = \sum_{x_j > 1} \phi_n(x_j)$ and $B = \sum_{x_j < 1} (-\phi_n(x_j))$, so $A > B$. Under the radial-order hypothesis, there is $r_0 > 0$ such that $\rho_n(x_i) \geq r_0$ on positive terms and $\rho_n(x_j) \leq r_0$ on negative terms. Therefore

$$S_{n+1}^m - S_n^m \geq r_0(A - B) = r_0 S_n^m > 0.$$

□

Thus a transient violation requires a genuinely interlaced radial configuration. The one-below theorem explains the exceptional rigidity of $m = 3$: by Theorem 3.4, every three-variable violation has exactly two coordinates above 1 and one below, so every violation lies in the persistent sector.

4 Exact pointwise exponent dynamics

Proposition 4.1 (Radial decomposition). *Fix $x \in \mathcal{H}_m$. For each $r \in (0, 1)$, let a_r be the number of coordinates equal to $1/r$ and b_r the number equal to r . Define*

$$L(x) = \sum_{x_j > 1} \frac{1}{x_j} - \#\{j : x_j < 1\}.$$

Then

$$S_n^m(x) = L(x) + \sum_r \frac{(1+r)r^n}{1+r^{n+1}} (b_r - a_r r), \quad (6)$$

where only finitely many radii occur.

Proof. If $0 < r < 1$, then

$$\phi_n(r) = -1 + \frac{(1+r)r^n}{1+r^{n+1}},$$

while

$$\phi_n(1/r) = r - \frac{(1+r)r^{n+1}}{1+r^{n+1}}.$$

Grouping coordinates with the same reciprocal radius gives (6). □

Theorem 4.2 (Complete eventual-sign rule). *For every fixed non-identity point $x \in \mathcal{H}_m$, the sign of $S_n^m(x)$ is eventually constant. More precisely:*

- (i) if $L(x) > 0$, then $S_n^m(x) > 0$ for all sufficiently large n ;
- (ii) if $L(x) < 0$, then $S_n^m(x) < 0$ for all sufficiently large n ;
- (iii) if $L(x) = 0$, let r_* be the largest radius for which $b_{r_*} - a_{r_*} r_* \neq 0$. Then the eventual sign equals the sign of $b_{r_*} - a_{r_*} r_*$.

In particular, every fixed point undergoes only finitely many membership changes in the sequence $\mathcal{I}_1^m, \mathcal{I}_2^m, \dots$.

Proof. The first two assertions follow immediately from Proposition 4.1. If $L = 0$, the term with largest radius r_* dominates exponentially all smaller radii.

It remains to show that some coefficient is nonzero away from the identity. If $b_r = a_r r$ for every occupied radius, then $b_r < a_r$ for each such r . But the product constraint in logarithmic form is

$$\sum_r (a_r - b_r) \log(1/r) = 0,$$

whose left side would then be strictly positive. This is impossible. □

Lemma 4.3 (Zero count for real exponential polynomials). *Let*

$$P(\nu) = \sum_{j=1}^q c_j e^{\mu_j \nu}, \quad \mu_1 < \cdots < \mu_q,$$

with all $c_j \neq 0$. Unless $P \equiv 0$, the function P has at most $q - 1$ real zeros counted with multiplicity.

Proof. This is the classical Chebyshev-system zero-count phenomenon for real exponentials [6]; the elementary induction is included to make the oscillation bound self-contained. For $q = 1$ there is no zero. For $q > 1$, multiply by $e^{-\mu_1 \nu}$ and differentiate. The derivative is an exponential polynomial with at most $q - 1$ terms. If the original function has Z real zeros counted with multiplicity, Rolle's theorem, with multiplicities included at repeated zeros, gives at least $Z - 1$ zeros of the derivative. Induction yields $Z - 1 \leq q - 2$. $\square$

Theorem 4.4 (Finite oscillation bound). *Fix a non-identity $x \in \mathcal{H}_m$ and suppose its coordinates use exactly k distinct reciprocal radii $r_1, \dots, r_k \in (0, 1)$. Then the membership sequence*

$$\mathbf{1}_{\{S_n^m(x) > 0\}}, \quad n = 1, 2, \dots,$$

changes value at most

$$\boxed{2^k - 2}$$

times. More sharply, if the subset sums of $\lambda_i = \log(1/r_i)$ that occur after collecting equal frequencies give q nonzero exponential terms, then at most $q - 2$ membership changes are possible.

Proof. Extend the radial decomposition to a real exponent $\nu \geq 0$:

$$S_\nu^m(x) = L(x) + \sum_{i=1}^k \frac{(1 + r_i)(b_i - a_i r_i)}{e^{\lambda_i \nu} + r_i}.$$

Multiplication by the positive denominator

$$D(\nu) = \prod_{i=1}^k (e^{\lambda_i \nu} + r_i)$$

produces a real exponential polynomial $P(\nu) = D(\nu)S_\nu^m(x)$. Its frequencies are subset sums of the λ_i , hence there are at most 2^k distinct frequencies. It is not the zero function by Theorem 4.2. Moreover $S_0^m(x) = 0$, so $P(0) = 0$. By Lemma 4.3, the total multiplicity of positive zeros is therefore at most $q - 2$.

Every change of the discrete positivity indicator between consecutive integers consumes at least one unit of positive zero multiplicity of the continuous function. If a zero lies strictly between the two integers this is the intermediate-value theorem; if the shared endpoint is responsible for changes on both adjacent intervals, the zero is tangential and has multiplicity at least two. Hence the number of membership changes is at most the positive-zero multiplicity, proving the claim. $\square$

Corollary 4.5 (Sign-count trichotomy). *Let r be the number of coordinates above 1 and s the number below 1 at a fixed non-identity point of $\mathcal{H}_m$.*

- (i) *If $r = 1$, violation is impossible by Theorem 3.4.*
- (ii) *If $s = 1$, every violation is strictly persistent by Theorem 3.5.*

- (iii) If $r \leq s$, then $S_n^m(x) < 0$ for all sufficiently large n ; hence every violation in such a sector is transient.
- (iv) The mixed regime $r > s \geq 2$ can support re-entrant behaviour; an exact example is given in Example 8.10.

Proof. Only (iii) requires comment. Since every coordinate above 1 has reciprocal strictly below 1,

$$L(x) = \sum_{x_j > 1} \frac{1}{x_j} - s < r - s \leq 0.$$

The inequality is strict, so Theorem 4.2 gives eventual negativity. $\square$

Corollary 4.6 (Exact one-below eventual envelope). *For $x_1, \dots, x_{m-1} > 1$ and $z = (\prod_i x_i)^{-1}$,*

$$(x_1, \dots, x_{m-1}, z) \in \mathcal{I}_n^m \text{ for some } n \iff \sum_{i=1}^{m-1} \frac{1}{x_i} > 1.$$

Proof. The right-hand side is exactly $L(x) > 0$, which gives eventual violation by Theorem 4.2. Conversely, if $L \leq 0$ and the point violated at some exponent, Theorem 3.5 would force all later values to stay above a fixed positive number, whereas Proposition 4.1 gives $S_n^m \rightarrow L \leq 0$. This is impossible. $\square$

Corollary 4.7 (Topology of the one-below eventual sector). *In a fixed one-below permutation sector, put $y_i = 1/x_i \in (0, 1)$ for $1 \leq i \leq m-1$. The set of points that violate for at least one exponent is carried diffeomorphically onto*

$$\mathcal{T}_{m-1} = \left\{ y \in (0, 1)^{m-1} : \sum_{i=1}^{m-1} y_i > 1 \right\}.$$

Thus the eventual one-below violation sector is open, convex in reciprocal coordinates, connected, and contractible. Moreover the increasing integer-exponent violation sets compactly exhaust it.

Proof. The set identity is Corollary 4.6. The reciprocal-coordinate region is the intersection of an open cube with an open half-space, hence is convex. If $K \subseteq \mathcal{T}_{m-1}$, the limiting margin $L = \sum_i y_i - 1$ is uniformly positive on K , while the radial decomposition converges uniformly on compact subsets. Thus K lies in one sufficiently large exponent and, by one-below persistence, in every later exponent. $\square$

For $m = 3$ this becomes especially explicit.

Corollary 4.8 (Exact three-variable union). *Up to permutation,*

$$\bigcup_{n \geq 1} \mathcal{I}_n^3 = \left\{ (x, y, (xy)^{-1}) : x, y > 1, \frac{1}{x} + \frac{1}{y} > 1 \right\}.$$

Equivalently, $(x-1)(y-1) < 1$. Under reciprocal coordinates $a = 1/x$, $b = 1/y$, one sign sector is the open triangle

$$0 < a, b < 1, \quad a + b > 1.$$

Corollary 4.9 (Compact exhaustion and quantitative onset). *Fix the sign sector $x, y > 1 > z$ with $xyz = 1$, and write*

$$\mathcal{E} = \left\{ (x, y, (xy)^{-1}) : \frac{1}{x} + \frac{1}{y} > 1 \right\}.$$

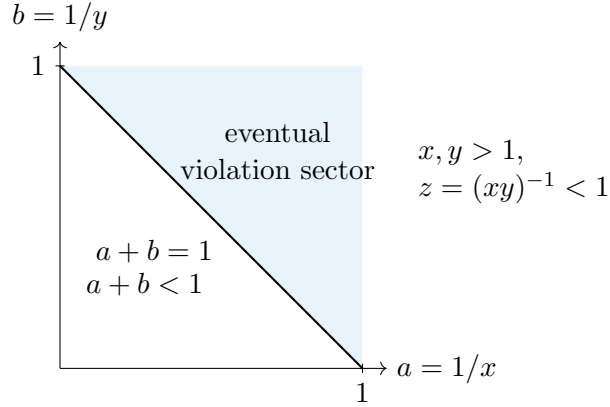


Figure 1. One of the three permutation sectors of the exact eventual three-variable violation envelope in reciprocal coordinates. The nested sets $\mathcal{I}_n^3$ compactly exhaust the shaded open triangle $a + b > 1$.

Then $\mathcal{I}_n^3 \cap \mathcal{E}$ increases to $\mathcal{E}$ as a compact exhaustion: every compact subset $K \subseteq \mathcal{E}$ is contained in $\mathcal{I}_N^3$ for some N , and hence in every $\mathcal{I}_n^3$ for $n \geq N$.

More explicitly, for a point of $\mathcal{E}$ set

$$L = \frac{1}{x} + \frac{1}{y} - 1 > 0.$$

It is sufficient that

$$n + 2 > \max \left\{ \frac{\log(2(x+1)/L)}{\log x}, \frac{\log(2(y+1)/L)}{\log y} \right\} \quad (7)$$

for the point to belong to $\mathcal{I}_n^3$.

Proof. For $t > 1$,

$$0 < \frac{1}{t} - \phi_n(t) = \frac{t+1}{t(t^{n+1}+1)} < \frac{t+1}{t^{n+2}},$$

while $\phi_n(z) > -1$ for $0 < z < 1$. Hence

$$S_n^3(x, y, z) > L - \frac{x+1}{x^{n+2}} - \frac{y+1}{y^{n+2}}.$$

Condition (7) makes each error smaller than $L/2$. The compact-exhaustion statement also follows abstractly from Corollary 4.8 and nestedness: the increasing open sets cover K , so compactness reduces the cover to finitely many exponents and the largest one contains all of K . $\square$

4.1 A fixed-exponent topology theorem

The eventual triangle describes the union over all exponents. At the first nonempty three-variable exponent one can also determine the exact component topology.

Theorem 4.10 (Topology of $\mathcal{I}_4^3$). *The set $\mathcal{I}_4^3$ has exactly three connected components, one for each choice of the coordinate below 1. Every component is contractible. More precisely, in the sector $x, y > 1 > z$ each fixed-product cross-section is an interval in logarithmic pair-spread, and the sector deformation-retracts onto the symmetric slice $x = y$.*

Proof. Fix the sector $x, y > 1 > z$, write

$$xy = t^2, \quad x + y = 2t + u(t-1)^2, \quad t > 1, 0 \leq u < 1.$$

The parameter $u = 0$ is the symmetric pair $x = y = t$, while $u \rightarrow 1$ is the boundary on which one of x, y tends to 1. Eliminating $z = t^{-2}$ gives a rational function $S_4(t, u)$ with strictly positive denominator.

On the symmetric slice $u = 0$, exact factorization gives

$$S_4(t, 0) = -\frac{(t-1)^2 P(t)}{(t^4 - t^3 + t^2 - t + 1)(t^8 - t^6 + t^4 - t^2 + 1)},$$

where

$$P(t) = t^{10} - t^9 - t^8 - t^7 + t^5 + t^4 + t^3 + t^2 + 2t + 2.$$

A Sturm sequence has sign-variation counts

$$6, 6, 5, 4, 4$$

at respectively

$$1, \quad \frac{37}{25}, \quad \frac{3}{2}, \quad \frac{8}{5}, \quad +\infty.$$

The exact sample signs used by the verifier are

$$P(1) = 6 > 0, \quad P\left(\frac{37}{25}\right) = \frac{6753216762174}{95367431640625} > 0,$$

$$P\left(\frac{3}{2}\right) = -\frac{217}{1024} < 0, \quad P\left(\frac{8}{5}\right) = \frac{3260034}{9765625} > 0.$$

Hence P has exactly two roots $a, b > 1$, with

$$\frac{37}{25} < a < \frac{3}{2} < b < \frac{8}{5},$$

and $S_4(t, 0) > 0$ exactly for $a < t < b$.

It remains to control pair-spread. Let $F(t, u)$ be the polynomial obtained from the cleared numerator so that $S_4(t, u) > 0$ iff $F(t, u) < 0$, and let $H(t, u)$ be the cleared derivative polynomial so that $\partial_u S_4(t, u) < 0$ iff $H(t, u) > 0$. Their bidegrees are $(14, 4)$ and $(22, 6)$. Compactify $t = (1 + s)/(1 - s)$, $0 \leq s < 1$. An exact two-variable Bernstein subdivision certificate proves

$$F(t, u) \leq 0 \implies H(t, u) > 0 \quad (t > 1, 0 \leq u < 1).$$

The certificate has only 9 nodes and 5 leaves: four leaves are excluded by $F > 0$ and the remaining leaf has $H > 0$. All coefficients and subdivisions are exact integers and are replayed by `damascus_topology_verify.py`.

Therefore, whenever a cross-section is nonnegative, increasing u strictly decreases S_4 . In particular, if $S_4(t, 0) \leq 0$, the cross-section can never enter the positive region: a first zero-to-positive crossing would have nonnegative right derivative, contradicting the certified strict negativity at every zero. At $u \rightarrow 1$ the limiting pair has one coordinate equal to 1, and the reciprocal identity gives a strictly negative value. Thus positive cross-sections occur exactly for $a < t < b$, and for each such t there is a unique threshold $U(t) \in (0, 1)$ such that

$$S_4(t, u) > 0 \iff 0 \leq u < U(t).$$

The implicit-function theorem and the strict derivative show that U is continuous. Write the pair explicitly as

$$x = te^\delta, \quad y = te^{-\delta}, \quad z = t^{-2}.$$

In these coordinates the cross-section parameter is

$$u(t, \delta) = \frac{x + y - 2t}{(t - 1)^2} = \frac{2t(\cosh \delta - 1)}{(t - 1)^2}. \quad (8)$$

For a violating point define

$$H_s(t, \delta) = \left(te^{(1-s)\delta}, te^{-(1-s)\delta}, t^{-2} \right), \quad 0 \leq s \leq 1.$$

Along H_s the product is fixed and $|(1-s)\delta|$ decreases monotonically to zero. Since $\cosh$ is strictly increasing on $[0, \infty)$, (8) gives

$$0 \leq u(t, (1-s)\delta) \leq u(t, \delta) < U(t).$$

Hence the certified u -monotonicity keeps $H_s(t, \delta)$ inside the same positive cross-section for every s . Moreover H_s fixes the symmetric slice pointwise when $\delta = 0$. Thus H is an explicit strong deformation retraction of the sector onto the interval $a < t < b$, and the sector is contractible.

By the three-variable sign-pattern theorem, every violation has exactly one coordinate below 1 and no coordinate equals 1. The identity of that coordinate cannot change along a path in $\mathcal{I}_4^3$. The three permutation sectors are nonempty and contractible, hence they are exactly the three connected components. $\square$

The fixed-product monotonicity in the preceding proof is not a formal consequence of three-variable nestedness. In fact, a new transverse phenomenon already occurs at exponent 7.

Proposition 4.11 (Exact off-axis fixed-product bifurcation). *There is a fixed-product cross-section of $\mathcal{I}_7^3$ whose symmetric point is nonviolating but whose off-axis part contains violating points. Explicitly, put*

$$t = \frac{79}{40}, \quad p = t^2, \quad q = \frac{397}{100}, \quad z = p^{-1},$$

and let x, y be the two roots of

$$T^2 - qT + p = 0.$$

Then $x, y > 1$, $xy = p$, and

$$S_7^3(t, t, z) < 0 < S_7^3(x, y, z).$$

Consequently the strong deformation retraction used for $\mathcal{I}_4^3$ cannot be extended to all exponents by fixed-product pair symmetrization.

Proof. The inequalities $q^2 > 4p$ and $q < p + 1$ imply that the roots are distinct and both exceed 1. For $P_0(q) = 2$, $P_1(q) = q$, and

$$P_k(q) = qP_{k-1}(q) - pP_{k-2}(q),$$

one has $P_k(q) = x^k + y^k$. The symmetric two-root reduction from the proof of Proposition 8.2 therefore evaluates both sums using rational arithmetic only. Exact cross-multiplication gives the stronger readable bounds

$$S_7^3(t, t, z) < -\frac{1}{5000} < 0 < \frac{1}{10000} < S_7^3(x, y, z).$$

The accompanying exact replay computes the reduced fractions directly from the recurrence and verifies both bounds. $\square$

5 Continuous exponents and a continuous three-variable filtration

The discrete exponent admits a natural real interpolation. For $\nu > 0$ define

$$\phi_\nu(t) = \frac{t^\nu - 1}{t^{\nu+1} + 1}, \quad S_\nu^m(x) = \sum_{j=1}^m \phi_\nu(x_j), \quad \mathcal{I}_\nu^m = \{x \in \mathcal{H}_m : S_\nu^m(x) > 0\}.$$

The reciprocal identity remains $\phi_\nu(1/t) = -t\phi_\nu(t)$.

Lemma 5.1 (Continuous radial multiplier). *For $\nu > 0$ and $t \neq 1$ set*

$$\kappa_\nu(t) = \frac{\partial_\nu \phi_\nu(t)}{\phi_\nu(t)},$$

and extend continuously by $\kappa_\nu(1) = 1/\nu$. Then $\kappa_\nu(t) > 0$, $\kappa_\nu(1/t) = \kappa_\nu(t)$, and κ_ν is strictly decreasing on $(1, \infty)$.

Proof. Write $t = e^u$, $u > 0$. The hyperbolic representation

$$\phi_\nu(e^u) = e^{-u/2} \frac{\sinh(\nu u/2)}{\cosh((\nu+1)u/2)}$$

gives

$$\kappa_\nu(e^u) = \frac{u \cosh(u/2)}{2 \sinh(\nu u/2) \cosh((\nu+1)u/2)}. \quad (9)$$

Positivity and the limit $u \rightarrow 0$ are immediate. Reciprocal invariance also follows by differentiating $\phi_\nu(1/t) = -t\phi_\nu(t)$ with respect to ν .

Taking the logarithmic derivative of (9) with respect to u yields

$$\frac{1}{u} + \frac{1}{2} \tanh \frac{u}{2} - \frac{\nu}{2} \coth \frac{\nu u}{2} - \frac{\nu+1}{2} \tanh \frac{(\nu+1)u}{2}.$$

Since $\coth y > 1/y$ for $y > 0$, the first and third terms have negative sum. Also

$$\frac{1}{2} \tanh \frac{u}{2} < \frac{\nu+1}{2} \tanh \frac{(\nu+1)u}{2}.$$

Thus the displayed logarithmic derivative is strictly negative. $\square$

Theorem 5.2 (Continuous one-below persistence). *Suppose x lies in a strict one-below sector: exactly one coordinate lies below 1 and every other coordinate lies above 1. If $S_\nu^m(x) > 0$ for some real $\nu > 0$, then*

$$\partial_\nu S_\nu^m(x) > \kappa_\nu(z) S_\nu^m(x) > 0,$$

where $z < 1$ is the unique coordinate below 1. Consequently $S_\mu^m(x) > S_\nu^m(x) > 0$ for every $\mu > \nu$.

Proof. After relabelling, let $x_1, \dots, x_{m-1} > 1 > z$ and put $a_i = \phi_\nu(x_i) > 0$, $c = -\phi_\nu(z) > 0$. Since

$$|\log z| = \sum_i \log x_i > \log x_i,$$

Lemma 5.1 gives $\kappa_\nu(x_i) > \kappa_\nu(z)$ for every i . Therefore

$$\partial_\nu S_\nu^m = \sum_i \kappa_\nu(x_i) a_i - \kappa_\nu(z) c > \kappa_\nu(z) \left(\sum_i a_i - c \right) = \kappa_\nu(z) S_\nu^m > 0.$$

The differential inequality prevents a positive value from returning to zero and yields strict increase thereafter. $\square$

Proposition 5.3 (Real-exponent one-above obstruction). *For every $\nu > 0$, if exactly one coordinate of $x \in \mathcal{H}_m$ lies above 1 and every other coordinate lies below 1, then*

$$S_\nu^m(x) < 0.$$

Proof. Use

$$\tau_\nu(t) = \frac{t^\nu - 1}{t^\nu + 1}.$$

The two strict comparisons in Lemma 3.1 remain valid with n replaced by ν , and for $a, b > 1$ the identity proving strict subadditivity depends only on $A = a^\nu > 1$ and $B = b^\nu > 1$. The proof of Theorem 3.4 therefore applies without change. $\square$

Corollary 5.4 (Continuous three-variable filtration and unique onset). *For every $0 < \nu < \mu$,*

$$\boxed{\mathcal{I}_\nu^3 \subseteq \mathcal{I}_\mu^3}.$$

For a fixed three-variable point in the one-below sector, positivity for some real exponent occurs if and only if

$$\frac{1}{x} + \frac{1}{y} > 1.$$

In that case there is a unique onset parameter $\nu_*(x, y, z) \geq 3$ such that

$$S_\nu^3 \leq 0 \quad (0 < \nu \leq \nu_*), \quad S_\nu^3 > 0 \quad (\nu > \nu_*),$$

and S_ν^3 is strictly increasing after onset.

Proof. Every three-variable violation has exactly two coordinates above 1 and one below, by Proposition 5.3 together with the product constraint and the reciprocal-pair calculation on a coordinate plane. Apply Theorem 5.2. If the eventual limit $L = 1/x + 1/y - 1$ is nonpositive, any positive value would persist and contradict $S_\nu^3 \rightarrow L$ as $\nu \rightarrow \infty$; if $L > 0$, positivity holds for all sufficiently large ν . Continuity then makes the positivity set one interval (ν_*, ∞) . The original inequality is valid at exponent 3, so persistence forces $\nu_* \geq 3$. $\square$

The preceding corollary gives a pointwise onset. The first parameter at which the set can become nonempty is also explicit. The key observation is that, although fixed-product symmetrization fails at larger exponents (Proposition 4.11), every positive global maximum is nevertheless symmetric.

Lemma 5.5 (Symmetry of positive three-variable maxima). *Fix $\nu > 0$. In a sign sector $x, y > 1 > z$ with $xyz = 1$, every positive global maximum of S_ν^3 satisfies $x = y$.*

Proof. Write $z = (xy)^{-1}$ and use reciprocal coordinates $a = 1/x$, $b = 1/y$. The sum extends continuously to the compact square $[0, 1]^2$. On $a = 1$ or $b = 1$ it reduces to a reciprocal pair and is nonpositive, while on $a = 0$ or $b = 0$ it is strictly negative. Hence, if a positive value exists, the global maximum is positive and is attained in the interior $0 < a, b < 1$, equivalently at logarithmic coordinates $x = e^u$, $y = e^v$ with $u, v > 0$.

Put

$$h_\nu(t) = t\phi'_\nu(t) = \frac{t^\nu\{\nu + (\nu + 1)t - t^{\nu+1}\}}{(1 + t^{\nu+1})^2}.$$

The critical-point equations in (u, v) give

$$h_\nu(x) = h_\nu(y) = h_\nu(z).$$

The last quantity is positive because $0 < z < 1$. Consequently x and y lie between 1 and the unique zero $\alpha_\nu > 1$ of h_ν on $(1, \infty)$.

It remains to note that h_ν is strictly decreasing on $(1, \alpha_\nu)$. Set $X = t^{\nu+1}$ and $C = \nu + (\nu+1)t$; on this interval $1 < X < C$. Direct differentiation gives

$$h'_\nu(t) = -\frac{XB_\nu(t, X)}{t^2(1+X)^3},$$

where

$$B_\nu(t, X) = -X^2 + \{\nu^2 t + \nu^2 + 2\nu t + 4\nu + t + 1\}X - \nu^2 t - \nu^2 - 2\nu t - t.$$

As a function of X , this is concave. Its endpoint values on $[1, C]$ are

$$B_\nu(t, 1) = 4\nu, \quad B_\nu(t, C) = \nu(\nu+1)^2(t+1)^2,$$

both strictly positive. Hence $h'_\nu < 0$ on the required interval, and the equalities $h_\nu(x) = h_\nu(y)$ force $x = y$. $\square$

For $1 < t < 2$ define

$$p_+(t) = \frac{t^2 + t + \sqrt{t^4 - 2t^3 + 9t^2 + 8t - 16}}{2t(2-t)}, \quad \eta(t) = \frac{\log p_+(t)}{\log t}, \quad (10)$$

and put

$$\boxed{\nu_c = \min_{1 < t < 2} \eta(t)}. \quad (11)$$

The minimum exists because, as $s \downarrow 0$,

$$p_+(1+s) = 1 + \sqrt{6}s + O(s), \quad p_+(2-s) = \frac{3}{s} + O(1),$$

so $\eta(t) \rightarrow +\infty$ at both endpoints.

Theorem 5.6 (Exact continuous three-variable onset). *For real $\nu > 0$,*

$$\boxed{\mathcal{I}_\nu^3 \neq \emptyset \iff \nu > \nu_c}.$$

Numerically,

$$\nu_c = 3.9826231561383400589\dots, \quad t_c = 1.5355671257762481179\dots,$$

where t_c is a minimizer in (11). The numerical values are descriptive; the exact definition is (11).

Proof. By Proposition 5.3, a three-variable violation has, after permutation, two coordinates above 1 and one below. If the set is nonempty, S_ν^3 has a positive global maximum in that sector, and Lemma 5.5 puts it on (t, t, t^{-2}) .

Set $p = t^\nu$. Exact simplification gives

$$S_\nu^3(t, t, t^{-2}) = -\frac{(p-1)F(t, p)}{(pt+1)(p^2t^2+1)}, \quad (12)$$

where

$$F(t, p) = p^2t^2(t-2) + pt^2(t+1) + t^2 - 2.$$

For $t \geq 2$ this is positive, so the symmetric sum is negative. For $1 < t < 2$, the quadratic $F(t, p)$ starts positive at $p = 1$, tends to $-\infty$, and has exactly one root greater than 1. The quadratic formula identifies that root as $p_+(t)$ in (10). It follows from (12) that

$$S_\nu^3(t, t, t^{-2}) > 0 \iff t^\nu > p_+(t) \iff \nu > \eta(t).$$

Taking the union over $1 < t < 2$ proves the equivalence with $\nu > \min \eta$. At equality all symmetric values are nonpositive, and the maximum-symmetry lemma rules out an off-axis positive value. $\square$

The continuous filtration is already false in four variables at a half-integer parameter, and the failure can be checked without numerical approximation.

Proposition 5.7 (Exact half-integer four-variable failure). *One has*

$$\boxed{\mathcal{I}_{15/2}^4 \not\subseteq \mathcal{I}_8^4.}$$

An exact witness is obtained by taking

$$(r_1, r_2, r_3, r_4) = \left(\frac{127}{100}, \frac{32}{25}, \frac{63}{100}, \frac{15625}{16002} \right), \quad X = (r_1^2, r_2^2, r_3^2, r_4^2).$$

Then $\prod_j X_j = 1$ and $S_{15/2}^4(X) > 0 > S_8^4(X)$.

Proof. The product identity follows from $\prod_j r_j = 1$. For every positive rational r ,

$$\phi_{15/2}(r^2) = \frac{r^{15} - 1}{r^{17} + 1},$$

so both signs are rational computations. Exact cross-multiplication yields

$$S_{15/2}^4(X) > \frac{1}{20000} > 0, \quad S_8^4(X) < -\frac{1}{1000} < 0.$$

The reduced fractions and both comparisons are independently regenerated by the exact replay script. $\square$

The isolated half-integer witness is part of a general obstruction: once a three-variable violation exists, a four-variable transient point can be created arbitrarily close to its coordinate trace.

Theorem 5.8 (No infinite forward inclusion ray beyond onset). *For every real $\nu > \nu_c$, there is $\mu_0 > \nu$ such that*

$$\mathcal{I}_\nu^4 \not\subseteq \mathcal{I}_\mu^4 \quad \text{for every } \mu \geq \mu_0.$$

The hypothesis $\nu > \nu_c$ is, by Theorem 5.6, equivalent to nonemptiness of $\mathcal{I}_\nu^3$.

Proof. Choose $(x, y, z) \in \mathcal{I}_\nu^3$. By the real one-above obstruction and the product constraint, after permutation one has $x, y > 1 > z$ and $xyz = 1$. For $r < 1$ sufficiently close to 1, put

$$X_r = (x/r, y, z, r).$$

Then $X_r \in \mathcal{H}_4$, it lies in a strict $2 + 2$ sector, and continuity gives

$$S_\nu^4(X_r) \longrightarrow S_\nu^3(x, y, z) > 0 \quad (r \uparrow 1).$$

Thus fix such an r with $S_\nu^4(X_r) > 0$. Its eventual-sign constant from Proposition 4.1 is

$$L(X_r) = \frac{r}{x} + \frac{1}{y} - 2 < 0,$$

because both positive coordinates x/r and y exceed 1, while there are two coordinates below 1. The radial decomposition, valid for real exponents as in the proof of Theorem 4.4, gives

$$S_\mu^4(X_r) \longrightarrow L(X_r) < 0 \quad (\mu \rightarrow \infty).$$

Hence $S_\mu^4(X_r) < 0$ for every sufficiently large μ , proving the general statement. The equivalence with $\nu > \nu_c$ is Theorem 5.6. $\square$

6 Escape energy and complete boundedness

Lemma 6.1 (One-variable maximum). *The function ϕ_n has a unique global maximum at $\alpha_n > 1$, where*

$$\alpha_n^{n+1} = (n+1)\alpha_n + n. \quad (13)$$

The maximum value is

$$M_n := \phi_n(\alpha_n) = \frac{n}{(n+1)\alpha_n}. \quad (14)$$

Moreover M_n is strictly increasing in n .

Proof. Differentiation gives

$$\phi'_n(t) = \frac{t^{n-1}[n + (n+1)t - t^{n+1}]}{(t^{n+1} + 1)^2}.$$

The bracket is strictly decreasing after its unique root above 1, giving uniqueness. Substitution of (13) into ϕ_n gives (14). Finally $\phi_{n+1}(t) - \phi_n(t) > 0$ for every $t > 1$, so maximizing implies $M_{n+1} > M_n$. $\square$

Lemma 6.2 (Limit of the one-variable maxima). *The sequence (M_n) increases strictly to 1.*

Proof. Strict increase was proved in Lemma 6.1. For $t > 1$,

$$0 < \phi_n(t) < \frac{1}{t} < 1,$$

so $M_n < 1$. Conversely, for every fixed $t > 1$,

$$M_n \geq \phi_n(t) \longrightarrow \frac{1}{t}.$$

Taking $n \rightarrow \infty$ and then $t \downarrow 1$ gives $\liminf M_n \geq 1$. $\square$

Definition 6.3 (Escape level). For $R > 0$ set

$$E_n^m(R) = \sup \left\{ S_n^m(x) : x \in \mathcal{H}_m, \max_j |\log x_j| \geq R \right\}.$$

The function $R \mapsto E_n^m(R)$ is nonincreasing and bounded, since $-1 < \phi_n(t) < 1$ for $t > 0$. Hence the finite limit exists, and we define

$$\Lambda_n^m = \lim_{R \rightarrow \infty} E_n^m(R).$$

Theorem 6.4 (Exact escape level). *For all $m \geq 2$ and $n \geq 1$,*

$$\boxed{\Lambda_n^m = -1 + (m-2)M_n.} \quad (15)$$

Proof. Take any escaping sequence and pass to a subsequence for which each coordinate converges in the extended interval $[0, \infty]$. The product constraint forces at least one coordinate to tend to 0 and at least one to tend to ∞ . Since

$$\phi_n(t) \rightarrow -1 \quad (t \rightarrow 0^+), \quad \phi_n(t) \rightarrow 0 \quad (t \rightarrow \infty),$$

and every remaining coordinate contributes at most M_n , we obtain

$$\limsup S_n^m \leq -1 + (m-2)M_n.$$

For sharpness use

$$X_\varepsilon = \left(\varepsilon, \frac{1}{\varepsilon \alpha_n^{m-2}}, \alpha_n, \dots, \alpha_n \right).$$

Its product is 1, and as $\varepsilon \rightarrow 0^+$,

$$S_n^m(X_\varepsilon) \rightarrow -1 + (m-2)M_n.$$

□

Proposition 6.5 (Exact sign-sector escape energy). *Fix integers $r, s \geq 1$ with $r + s \leq m$, and consider the stratum of $\mathcal{H}_m$ having exactly r coordinates above 1, exactly s below 1, and the remaining coordinates equal to 1. Its escape level is*

$$\boxed{\Lambda_n^{(r,s)} = -1 + (r-1)M_n.} \quad (16)$$

Consequently the violation set inside this stratum is bounded if and only if

$$(r-1)M_n < 1. \quad (17)$$

Proof. Along any escaping sequence in the stratum, the logarithmic product constraint forces at least one coordinate above 1 to tend to ∞ and at least one coordinate below 1 to tend to 0. Their limiting contributions are 0 and -1 , respectively. The other $r-1$ positive coordinates contribute at most M_n , while the remaining negative coordinates contribute at most 0. Hence

$$\limsup S_n^m \leq -1 + (r-1)M_n.$$

For sharpness, let $\delta = \delta_\varepsilon \downarrow 0$ and take, up to permutation,

$$\varepsilon, \quad e^{-\delta} \text{ (} s-1 \text{ times)}, \quad \alpha_n \text{ (} r-1 \text{ times)}, \quad P_\varepsilon := \frac{e^{(s-1)\delta}}{\varepsilon \alpha_n^{r-1}},$$

with all remaining coordinates equal to 1. Their product is 1, $P_\varepsilon \rightarrow \infty$, and the sum tends to the right-hand side of (16).

If the escape level is positive, these points are violating for all small ε . In the critical case $(r-1)M_n = 1$, choose for instance $\delta_\varepsilon = \varepsilon^{n+2}$. Then

$$S_n^m = (1 + \phi_n(\varepsilon)) + \phi_n(P_\varepsilon) + (s-1)\phi_n(e^{-\delta_\varepsilon}),$$

where the first two terms are positive, $1 + \phi_n(\varepsilon) \asymp \varepsilon^n$, and the negative final term is $O(\varepsilon^{n+2})$. Hence violations still escape. If the escape level is negative, the defining limsup excludes every sufficiently distant violation. This proves (17). □

Corollary 6.6 (Complete sign-sector confinement diagram). *For a sign stratum with $r \geq 2$ coordinates above 1, boundedness depends only on r and n as follows:*

r	bounded exponents	critical equality
2	every n	none
3	$n \leq 4$	none
4	$n = 1$	$n = 2$
5	$n = 1$	none
$r \geq 6$	none	none

In particular, the only integer critical sector is $(r, n) = (4, 2)$.

Proof. Apply $(r-1)M_n < 1$ from Proposition 6.5. For the critical equality, note

$$\frac{1}{M_1} = 2(1 + \sqrt{2}) \in (4, 5), \quad \frac{1}{M_2} = 3.$$

Since $M_2 < M_3 < M_4 < 1/2$, one has $2 < 1/M_n < 3$ for $n = 3, 4$. Since $M_n > 1/2$ for every $n \geq 5$ while $M_n < 1$, one has $1 < 1/M_n < 2$ there. Thus the only integral reciprocal is $1/M_2 = 3$, yielding $(r, n) = (4, 2)$. The bounded rows follow from the same inequalities and monotonicity. $\square$

Corollary 6.7 (Bounded transient islands in four variables). *For every $n \geq 1$, the portion of $\mathcal{I}_n^4$ with two coordinates above 1 and two below 1 is bounded.*

Proof. Here $r = s = 2$, so Proposition 6.5 gives escape level $-1 + M_n < 0$, since $\phi_n(t) < 1/t < 1$ for $t > 1$ and therefore $M_n < 1$. $\square$

Theorem 6.8 (Complete boundedness criterion). *The violation set $\mathcal{I}_n^m$ is bounded if and only if*

$$(m-2)M_n < 1. \tag{18}$$

If equality holds, the violation set is already unbounded.

Proof. If (18) holds, Theorem 6.4 rules out an escaping violating sequence. Conversely, if $(m-2)M_n > 1$, the family X_ε from Theorem 6.4 is violating for all sufficiently small ε . If equality holds, then

$$S_n^m(X_\varepsilon) = (\phi_n(\varepsilon) + 1) + \phi_n\left(\frac{1}{\varepsilon \alpha_n^{m-2}}\right) > 0$$

for sufficiently small ε , so again the violation set is unbounded. $\square$

Corollary 6.9 (Integer boundedness phase diagram). *The complete boundedness classification is:*

- $m \leq 3$: bounded for every n ;
- $m = 4$: bounded exactly for $n \leq 4$;
- $m = 5, 6$: bounded exactly for $n = 1$;
- $m \geq 7$: unbounded for every n .

Among nonempty violation sets, the bounded cases are precisely

$$m = 3, n \geq 4; \quad m = 4, n = 2, 3, 4; \quad (m, n) = (6, 1).$$

Proof. We have

$$M_1 = \frac{\sqrt{2}-1}{2}, \quad M_2 = \frac{1}{3},$$

while $M_4 < 1/2 < M_5$. The latter inequalities follow by evaluating the strictly increasing critical equations $t^5 - 5t - 4$ at $8/5$ and $t^6 - 6t - 5$ at $5/3$. Since M_n increases in n , Theorem 6.8 gives the table. The nonempty subtable follows from Table 1. $\square$

Corollary 6.10 (Unique critical lattice point). *The equality $\Lambda_n^m = 0$ occurs for exactly one integer pair:*

$$(m, n) = (5, 2).$$

For this pair,

$$X_\varepsilon = \left(\varepsilon, \frac{1}{8\varepsilon}, 2, 2, 2 \right)$$

satisfies

$$S_2^5(X_\varepsilon) = \frac{\varepsilon(64\varepsilon^2 - 71\varepsilon + 8)}{(\varepsilon^2 - \varepsilon + 1)(64\varepsilon^2 - 8\varepsilon + 1)}. \quad (19)$$

Hence $X_\varepsilon \in \mathcal{I}_2^5$ for

$$0 < \varepsilon < \frac{71 - \sqrt{2993}}{128},$$

and $S_2^5(X_\varepsilon) \rightarrow 0^+$ as $\varepsilon \rightarrow 0^+$.

Proof. Criticality requires $m - 2 = 1/M_n$ to be an integer. At $n = 1$,

$$\frac{1}{M_1} = 2(1 + \sqrt{2}) \in (4, 5),$$

so no integer m occurs. At $n = 2$, $1/M_2 = 3$, giving $m = 5$. For $n = 3, 4$, strict monotonicity and $M_2 = 1/3 < M_n < 1/2$ give

$$2 < \frac{1}{M_n} < 3,$$

while for every $n \geq 5$, $1/2 < M_n < 1$ gives

$$1 < \frac{1}{M_n} < 2.$$

Hence $(5, 2)$ is the unique lattice point. Formula (19) is direct algebra. $\square$

Proposition 6.11 (Second-order stability at the critical escape profile). *Fix $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. For sufficiently small $\varepsilon > 0$ define*

$$X_\varepsilon(y) = \left(\varepsilon, \frac{1}{\varepsilon \prod_{j=1}^3 (2 + \sqrt{\varepsilon} y_j)}, 2 + \sqrt{\varepsilon} y_1, 2 + \sqrt{\varepsilon} y_2, 2 + \sqrt{\varepsilon} y_3 \right) \in \mathcal{H}_5.$$

Then

$$\frac{S_2^5(X_\varepsilon(y))}{\varepsilon} \rightarrow 8 - \frac{1}{9} \|y\|_2^2. \quad (20)$$

The convergence is uniform for y in compact subsets of $\mathbb{R}^3$. Hence every fixed perturbation with $\|y\|_2^2 < 72$ remains on the violating side for all sufficiently small ε , whereas every fixed perturbation with $\|y\|_2^2 > 72$ lies on the nonviolating side. Thus the critical escape tube has the second-order stability threshold $\|y\|_2 = 6\sqrt{2}$.

Proof. For $\phi_2(t) = (t^2 - 1)/(t^3 + 1)$,

$$\phi_2(2) = \frac{1}{3}, \quad \phi_2'(2) = 0, \quad \phi_2''(2) = -\frac{2}{9}.$$

Therefore, uniformly for bounded y ,

$$\sum_{j=1}^3 \phi_2(2 + \sqrt{\varepsilon} y_j) = 1 - \frac{\varepsilon}{9} \sum_{j=1}^3 y_j^2 + O(\varepsilon^{3/2}).$$

Also $\phi_2(\varepsilon) = -1 + O(\varepsilon^2)$. If

$$P_\varepsilon = \frac{1}{\varepsilon \prod_j (2 + \sqrt{\varepsilon} y_j)},$$

then $P_\varepsilon \rightarrow \infty$ and

$$\phi_2(P_\varepsilon) = \frac{1}{P_\varepsilon} + O(P_\varepsilon^{-3}) = 8\varepsilon + O(\varepsilon^{3/2}).$$

Adding the three expansions proves (20). $\square$

Proposition 6.12 (Exact classification on the critical stability sphere). *Assume $\|y\|_2^2 = 72$ and define*

$$\mathcal{C}(y) = \sum_{j=1}^3 (y_j^3 + 36y_j).$$

Then, for fixed y as $\varepsilon \rightarrow 0^+$,

$$S_2^5(X_\varepsilon(y)) = \frac{\varepsilon^{3/2}}{9} \mathcal{C}(y) + \varepsilon^2 \mathcal{Q}(y) + O(\varepsilon^{5/2}), \quad (21)$$

$$\mathcal{Q}(y) = 1 + 2 \sum_{1 \leq i < j \leq 3} y_i y_j - \frac{2}{27} \sum_{j=1}^3 y_j^4. \quad (22)$$

Consequently:

- (i) if $\mathcal{C}(y) > 0$, then $X_\varepsilon(y) \in \mathcal{I}_2^5$ for every sufficiently small $\varepsilon > 0$;
- (ii) if $\mathcal{C}(y) < 0$, then $X_\varepsilon(y) \notin \mathcal{I}_2^5$ for every sufficiently small $\varepsilon > 0$;
- (iii) if $\mathcal{C}(y) = 0$, then again $X_\varepsilon(y) \notin \mathcal{I}_2^5$ for every sufficiently small $\varepsilon > 0$.

Thus all fixed transverse directions on the second-order sphere $\|y\|_2 = 6\sqrt{2}$ are classified: the cubic $\mathcal{C}$ decides the generic directions, and its zero locus lies strictly on the nonviolating side at quartic order. The statement does not claim a uniform classification of arbitrary ε -dependent perturbations that drift toward the cubic zero locus.

Proof. Put $h = \sqrt{\varepsilon}$. Direct Taylor expansion of the exact rational expression gives

$$S_2^5(X_{h^2}(y)) = h^2 \left(8 - \frac{1}{9} \sum_j y_j^2 \right) + \frac{h^3}{9} \mathcal{C}(y) + h^4 \mathcal{Q}(y) + O(h^5),$$

uniformly for y in compact sets. On $\sum_j y_j^2 = 72$, the quadratic coefficient vanishes and (21) follows.

It remains only to settle the case $\mathcal{C}(y) = 0$. Write $s_1 = y_1 + y_2 + y_3$, $p_k = \sum_j y_j^k$. Since $p_2 = 72$,

$$2 \sum_{i < j} y_i y_j = s_1^2 - 72, \quad \mathcal{Q}(y) = s_1^2 - 71 - \frac{2}{27} p_4.$$

The equation $\mathcal{C}(y) = 0$ is $p_3 = -36s_1$. By Cauchy–Schwarz,

$$p_4 p_2 \geq p_3^2,$$

so $p_4 \geq 18s_1^2$. Therefore

$$\mathcal{Q}(y) \leq s_1^2 - 71 - \frac{36}{27} s_1^2 = -71 - \frac{1}{3} s_1^2 < 0.$$

This gives strict negativity at quartic order on the cubic zero locus and completes the classification. $\square$

Proposition 6.13 (Continuous confinement curves). *For every real $\nu > 0$, let*

$$M(\nu) = \max_{t>0} \phi_\nu(t).$$

There is a unique maximizer $\alpha_\nu > 1$ satisfying

$$\alpha_\nu^{\nu+1} = (\nu+1)\alpha_\nu + \nu, \quad M(\nu) = \frac{\nu}{(\nu+1)\alpha_\nu}.$$

The function $M : (0, \infty) \rightarrow (0, 1)$ is continuous and strictly increasing, with

$$M(\nu) \rightarrow 0 \quad (\nu \downarrow 0), \quad M(\nu) \rightarrow 1 \quad (\nu \rightarrow \infty).$$

For real exponent ν , the exact global and sign-sector escape energies are

$$\boxed{\Lambda_\nu^m = -1 + (m-2)M(\nu)}, \quad \boxed{\Lambda_\nu^{(r,s)} = -1 + (r-1)M(\nu)}.$$

Consequently, for every $m \geq 4$ there is a unique critical real exponent ν_m determined by

$$M(\nu_m) = \frac{1}{m-2}.$$

For fixed $m \geq 4$, the real-exponent violation set is bounded for $0 < \nu < \nu_m$ and unbounded for $\nu \geq \nu_m$; equality is already critical-unbounded. Specifically,

$$\nu_4 \in (4, 5), \quad \nu_5 = 2, \quad \nu_6 \in (1, 2), \quad 0 < \nu_m < 1 \quad (m \geq 7),$$

while $m = 3$ has no finite critical exponent and is critical only in the limit $\nu \rightarrow \infty$.

Proof. Differentiation in t gives the same critical numerator as in Lemma 6.1, with n replaced by ν :

$$t^{\nu-1}[\nu + (\nu+1)t - t^{\nu+1}].$$

The bracket is positive at $t = 1$, strictly decreasing for $t > 1$, and tends to $-\infty$, hence has one root. Substitution gives the formula for $M(\nu)$. The implicit-function theorem gives continuity of α_ν and hence of M . For fixed $t > 1$, Lemma 5.1 implies $\partial_\nu \phi_\nu(t) > 0$, so maximization gives strict increase of M .

The formula $M(\nu) = \nu/((\nu+1)\alpha_\nu)$ gives $0 < M(\nu) < \nu/(\nu+1)$ and therefore the limit 0 at the left endpoint. At the right endpoint, $M(\nu) \geq \phi_\nu(t) \rightarrow 1/t$ for each fixed $t > 1$; then let $t \downarrow 1$. The escape proofs in Theorem 6.4 and Proposition 6.5 use only the one-variable maximum and the limits of ϕ_ν at 0 and ∞ , so they carry over verbatim. At equality, the same perturbation used in Proposition 6.5, with $\delta_\varepsilon = \varepsilon^{\nu+2}$, gives escaping positive values because $1 + \phi_\nu(\varepsilon) \asymp \varepsilon^\nu$ while the added near-unit negative terms are $O(\varepsilon^{\nu+2})$. Thus critical equality is already unbounded. Strict monotonicity of M gives uniqueness of each critical exponent. The displayed intervals follow from $M_1 = (\sqrt{2}-1)/2$, $M_2 = 1/3$, and $M_4 < 1/2 < M_5$. $\square$

Proposition 6.14 (Rigidity of extremal escape). *If an escaping sequence satisfies $S_n^m \rightarrow \Lambda_n^m$, then after passing to a subsequence and permuting coordinates, exactly one coordinate tends to 0, exactly one tends to ∞ , and every remaining coordinate tends to α_n .*

Proof. Pass to a subsequence on which every coordinate converges in $[0, \infty]$, and let k and ℓ be the numbers tending to 0 and ∞ , respectively. The product constraint gives $k, \ell \geq 1$, while the same estimate as in Theorem 6.4 gives

$$\limsup S_n^m \leq -k + (m - k - \ell)M_n.$$

The difference between $\Lambda_n^m = -1 + (m-2)M_n$ and this upper bound is

$$(k-1) + (k+\ell-2)M_n,$$

which can vanish only when $k = \ell = 1$. Equality then also forces every remaining finite coordinate to attain the one-variable maximum M_n . Since α_n is the unique maximizer of ϕ_n , all those coordinates converge to α_n . $\square$

7 Coordinate traces and complete separation

Put $x_j = e^{u_j}$ and $g_n(u) = \phi_n(e^u)$. At $u = 0$,

$$g_n(u) = \frac{n}{2}u - \frac{n}{4}u^2 - \frac{n^2(n+3)}{24}u^3 + O(u^4). \quad (23)$$

Hence on $\sum_j u_j = 0$,

$$S_n^m = -\frac{n}{4} \sum_j u_j^2 + O(\|u\|^3). \quad (24)$$

Thus the identity is a strict local maximum of value zero for every m, n .

Proposition 7.1 (Coordinate-trace identity). *For every j ,*

$$\mathcal{I}_n^m \cap \{x_j = 1\} \cong \mathcal{I}_n^{m-1}.$$

More generally, fixing k coordinates equal to 1 yields an exact copy of $\mathcal{I}_n^{m-k}$.

Proof. Simply use $\phi_n(1) = 0$; the product constraint and the value of S_n reduce to the lower-dimensional ones. $\square$

Lemma 7.2 (Bounded trace criterion). *Suppose $\mathcal{I}_n^m$ is bounded, $\mathcal{I}_n^{m-1} = \emptyset$, and the zero set of S_n^{m-1} on $\mathcal{H}_{m-1}$ consists only of the identity. Then $\mathcal{I}_n^m$ stays a positive distance from every coordinate plane $x_j = 1$.*

Proof. If not, a bounded violating sequence would approach one such plane. Boundedness together with the product constraint also bounds all coordinates away from zero, so a subsequence converges in $\mathcal{H}_m$. The trace identity forces its lower-dimensional limit to satisfy $S_n^{m-1} = 0$, hence to be the identity. This contradicts the strict local negativity (24). $\square$

Theorem 7.3 (Complete separation classification). *Among nonempty violation sets, separation from all coordinate planes occurs exactly for*

$$m = 3, n \geq 4; \quad (m, n) = (4, 2), (4, 3); \quad (m, n) = (6, 1).$$

Every other nonempty $\mathcal{I}_n^m$ intersects a coordinate plane.

Proof. The $m = 3$ statement is Theorem 5.1 of Sujstintinukul–Chesneau [8]. For $(4, 2)$ and $(4, 3)$, boundedness follows from Corollary 6.9; the corresponding three-variable inequality is valid with equality only at the identity, so Lemma 7.2 applies. For $(6, 1)$, $\mathcal{I}_1^6$ is bounded while the five-variable inequality is valid with unique equality at the identity [8]; again apply Lemma 7.2.

In every remaining nonempty case, Table 1 shows that $\mathcal{I}_n^{m-1}$ is already nonempty, so Proposition 7.1 gives an actual intersection with $x_j = 1$. $\square$

Corollary 7.4 (Unique bounded/non-separated pair). *The pair $(m, n) = (4, 4)$ is the unique nonempty case for which $\mathcal{I}_n^m$ is bounded but not separated.*

8 Breakdown of exponent nesting beyond dimension three

Theorem 8.1 (Global dimension classification of nestedness). *The family $\{\mathcal{I}_n^m\}_{n \geq 1}$ is increasing for every exponent if and only if $m \leq 3$.*

Proof. For $m = 1, 2$, all violation sets are empty. For $m = 3$, a point with a coordinate equal to 1 reduces to a reciprocal pair and has nonpositive sum. Together with Theorems 3.4 and 3.5, this shows that every three-variable violation lies in the one-below sector and therefore persists strictly.

For $m = 4$, the exact rational point

$$X_7 = \left(\frac{70}{41}, \frac{30}{17}, \frac{12}{35}, \frac{697}{720} \right)$$

has product 1 and satisfies

$$S_7^4(X_7) > 0 > S_8^4(X_7).$$

Exact rational numerators are supplied in the accompanying verification script. Appending coordinates equal to 1 embeds the same failure into every $m > 4$. $\square$

The previous theorem only needs one failure. Before proving the permanent failure regime, we close the complementary low-exponent four-variable cases by an exact Bernstein certificate.

Proposition 8.2 (Four-variable persistence through exponent 6). *For $n = 1, \dots, 6$,*

$$\boxed{\mathcal{I}_n^4 \subseteq \mathcal{I}_{n+1}^4.}$$

Proof. The case $n = 1$ is immediate because $\mathcal{I}_1^4 = \emptyset$. By Theorem 3.4, a four-variable violation has either a $3 + 1$ or a $2 + 2$ sign pattern. Every $3 + 1$ violation persists by Theorem 3.5, so it remains only to treat the $2 + 2$ sector.

Write such a point as

$$x, y > 1, \quad a^{-1}, b^{-1} < 1, \quad xy = ab = p.$$

Put $q = x + y$ and $r = a + b$. By Corollary 3.3, violation forces $q < r$. Write $p = t^2$ with $t > 1$. Since a pair of numbers greater than 1 with product t^2 has sum in $[2t, t^2 + 1)$, there are unique parameters

$$0 \leq u < v < 1, \quad q = 2t + u(t - 1)^2, \quad r = 2t + v(t - 1)^2.$$

Let $P_0(Q) = 2$, $P_1(Q) = Q$, and

$$P_k(Q) = QP_{k-1}(Q) - pP_{k-2}(Q),$$

so that $P_k(q) = x^k + y^k$ and $P_k(r) = a^k + b^k$. Direct symmetric reduction gives

$$\begin{aligned} A_n(p, q) &:= \phi_n(x) + \phi_n(y) = \frac{p^n q + P_n(q) - P_{n+1}(q) - 2}{p^{n+1} + 1 + P_{n+1}(q)}, \\ C_n(p, r) &:= a\phi_n(a) + b\phi_n(b) = \frac{2p^{n+1} + P_{n+1}(r) - pP_n(r) - r}{p^{n+1} + 1 + P_{n+1}(r)}. \end{aligned}$$

Thus $S_n^4 = A_n - C_n$ is a rational function of (t, u, v) . Its two pair denominators are

$$p^{n+1} + 1 + P_{n+1}(q) = (x^{n+1} + 1)(y^{n+1} + 1) > 0$$

and

$$p^{n+1} + 1 + P_{n+1}(r) = (a^{n+1} + 1)(b^{n+1} + 1) > 0,$$

so clearing them does not alter signs. The resulting numerator has the exact factor $-(t - 1)^2$; denote the remaining polynomial by $F_n(t, u, v)$, so that

$$S_n^4 > 0 \iff F_n < 0.$$

Compactify the full domain by

$$t = \frac{1+s}{1-s}, \quad v = u + (1-u)w, \quad (s, u, w) \in [0, 1]^3.$$

The second map is onto the closed triangle $0 \leq u \leq v \leq 1$. If $d_t = \deg_t F_n$, multiplication by $(1-s)^{d_t}$ clears the compactification denominator and is strictly positive for $s < 1$; the boundary $s = 1$ is then included by continuity of the cleared polynomial. The exact degrees before and after compactification are

n	2	3	4	5	6	7
$\deg_{(t,u,v)} F_n$	(6, 2, 2)	(14, 4, 4)	(14, 4, 4)	(22, 6, 6)	(22, 6, 6)	(30, 8, 8)
$\deg_{(s,u,w)} \tilde{F}_n$	(6, 4, 2)	(14, 8, 4)	(14, 8, 4)	(22, 12, 6)	(22, 12, 6)	(30, 16, 8)

and $\tilde{F}_n$ has an exact tensor-product Bernstein representation on the unit cube.

For $n = 2$ and $n = 3$, all Bernstein coefficients of F_n are strictly positive: respectively 105 out of 105 and 675 out of 675. Hence the $2 + 2$ sector contains no violation at either exponent.

For $n = 4, 5, 6$, a counterexample to one-step persistence would require simultaneously

$$F_n < 0, \quad F_{n+1} \geq 0.$$

An exact dyadic Bernstein subdivision excludes this semialgebraic set. On each subbox, either every Bernstein coefficient of F_n is nonnegative, excluding $F_n < 0$, or every Bernstein coefficient of F_{n+1} is negative, forcing $F_{n+1} < 0$. A deterministic exact-integer replay terminates after

n	4	5	6
subdivision nodes	33	339	2591
maximum depth	5	7	9

with no unresolved box. Bernstein range enclosure and dyadic de Casteljau subdivision use the standard convex-hull and subdivision properties of the Bernstein basis [5, 3]; with integer-scaled coefficients they are exact algebraic operations, so this is a finite certificate, not a numerical approximation. The accompanying script reconstructs the polynomials and replays every subdivision using integer arithmetic. Therefore no $2 + 2$ counterexample exists for $n = 4, 5, 6$, completing the proof. $\square$

The failure is in fact permanent from exponent 7 onward.

Lemma 8.3 (Uniform four-variable tail estimates). *Let $N \geq 10$, $n = N - 1$, $L = \log(4/3)$,*

$$r = e^{-L/N} = (3/4)^{1/N}, \quad X_n = \left(\frac{16}{49r}, r, \frac{7}{4}, \frac{7}{4} \right).$$

Then

$$S_n^4(X_n) > \frac{1}{7} \left[\frac{3L}{N} - 22 \left(\frac{4}{7} \right)^N \right] > 0, \quad (25)$$

$$S_{n+1}^4(X_n) < \frac{211}{147} \left(\frac{64}{147} \right)^N - \frac{3L}{49(N+L)} < 0. \quad (26)$$

Proof. Put $a = 16/(49r)$. Since $r > 3/4$, one has

$$0 < a < \frac{64}{147} < 1, \quad 1 + a < \frac{211}{147}.$$

At exponent $n = N - 1$, $\phi_n(a) > -1$. For $t = 7/4$ the exact identity

$$\phi_n(t) = \frac{1}{t} - \frac{t+1}{t(t^N+1)}$$

gives

$$0 < \frac{4}{7} - \phi_n(7/4) < \frac{11}{7} \left(\frac{4}{7}\right)^N.$$

Also $r^N = 3/4$, hence

$$\phi_n(r) = \frac{3/r - 4}{7}.$$

Replacing the a -coordinate by its lower bound -1 , and the two $7/4$ coordinates by the preceding lower bounds, gives

$$S_n^4(X_n) > \frac{3}{7} \left(\frac{1}{r} - 1\right) - \frac{22}{7} \left(\frac{4}{7}\right)^N.$$

Since $1/r = e^{L/N} > 1 + L/N$, this is the first inequality in (25).

At exponent $n + 1 = N$,

$$\phi_N(r) = -\frac{1}{4+3r}, \quad \phi_N(7/4) < \frac{4}{7},$$

and

$$\phi_N(a) = -1 + \frac{(1+a)a^N}{1+a^{N+1}} < -1 + \frac{211}{147} \left(\frac{64}{147}\right)^N.$$

Therefore

$$S_N^4(X_n) < \frac{211}{147} \left(\frac{64}{147}\right)^N - \frac{3(1-r)}{7(4+3r)}.$$

Because $4 + 3r < 7$ and $1 - e^{-x} \geq x/(1+x)$ for $x > 0$,

$$\frac{3(1-r)}{7(4+3r)} > \frac{3L}{49(N+L)},$$

which proves the first inequality in (26).

It remains only to verify the scalar signs uniformly. The elementary estimate

$$L = \log(4/3) > \frac{2}{7}$$

follows from $\log(1+x) > 2x/(2+x)$ at $x = 1/3$. At $N = 10$,

$$\frac{3}{35} - 22 \left(\frac{4}{7}\right)^{10} = \frac{5717461}{1412376245} > 0.$$

Since $N(4/7)^N$ is strictly decreasing for $N \geq 10$, the positive inequality persists for all larger N . For the negative estimate,

$$\frac{3L}{49(N+L)} > \frac{6}{49(7N+2)}.$$

At $N = 10$,

$$\frac{211}{147} \left(\frac{64}{147}\right)^{10} < \frac{1}{588} = \frac{6}{49(7 \cdot 10 + 2)}.$$

Finally $(7N+2)(64/147)^N$ is strictly decreasing for $N \geq 10$, so the strict negative inequality also persists. $\square$

Theorem 8.4 (Failure at every step from $n = 7$ onward). *For every $m \geq 4$ and every $n \geq 7$,*

$$\mathcal{I}_n^m \not\subseteq \mathcal{I}_{n+1}^m.$$

Proof. It suffices to treat $m = 4$. The step $7 \rightarrow 8$ is witnessed by X_7 above. The step $8 \rightarrow 9$ is witnessed by

$$X_8 = \left(\frac{5}{3}, \frac{5}{3}, \frac{3}{8}, \frac{24}{25} \right),$$

for which exact arithmetic gives $S_8^4 > 0 > S_9^4$.

Now let $n \geq 9$ and put $N = n + 1$. With X_n as in Lemma 8.3, that lemma gives directly

$$S_n^4(X_n) > 0 > S_{n+1}^4(X_n).$$

Thus every step from $n = 9$ onward fails in four variables. Together with the exact $7 \rightarrow 8$ and $8 \rightarrow 9$ witnesses, this proves failure for every $n \geq 7$. Appending coordinates equal to 1 preserves all sums S_n , and therefore handles every $m > 4$. $\square$

Corollary 8.5 (Sharp four-variable nesting threshold). *For every integer $n \geq 1$,*

$$\boxed{\mathcal{I}_n^4 \subseteq \mathcal{I}_{n+1}^4 \iff n \leq 6.}$$

Thus $7 \rightarrow 8$ is the first exponent step at which four-variable nesting fails.

Proof. The inclusions through $6 \rightarrow 7$ are Proposition 8.2; failure for every $n \geq 7$ is Theorem 8.4. $\square$

Lemma 8.6 (Certified local exclusion for the early staircase). *Let $g_n(u) = \phi_n(e^u)$ and put*

$$y(x) = \frac{|x - 1|}{x + 1} = \tanh \frac{|\log x|}{2}.$$

For $n = 1, 2, 3$, respectively, if every coordinate satisfies

$$y(x_j) \leq \frac{1}{2}, \quad y(x_j) \leq \frac{1}{8}, \quad y(x_j) \leq \frac{1}{16},$$

then $S_n^m(x) \leq 0$, with equality only at $(1, \dots, 1)$.

Proof. At the origin,

$$g_n(0) = 0, \quad g'_n(0) = \frac{n}{2}, \quad g''_n(0) = -\frac{n}{2}.$$

Exact one-variable Bernstein certificates give the derivative bounds

$$|g_1'''(u)| \leq \frac{4}{3} \quad (|u| \leq \log 3),$$

$$|g_2'''(u)| \leq 6 \quad \left(|u| \leq \log \frac{9}{7} \right), \quad |g_3'''(u)| \leq 15 \quad \left(|u| \leq \log \frac{17}{15} \right).$$

For $n = 1$, after $t = e^u$ and $y = (t - 1)/(t + 1)$, the two inequalities $4/3 \pm g_1'''(u) \geq 0$ reduce to degree-eight rational numerators. One is globally Bernstein-positive on $[-1/2, 1/2]$; the other is certified by the four dyadic intervals

$$[-1/2, -1/4], \quad [-1/4, -1/8], \quad [-1/8, 0], \quad [0, 1/2].$$

For $n = 2, 3$ the corresponding rational numerators are Bernstein-positive on the full intervals $[-1/8, 1/8]$ and $[-1/16, 1/16]$. These exact coefficient checks are replayed in the supplement.

Taylor's theorem gives

$$g_n(u) \leq \frac{n}{2}u - \frac{n}{4}u^2 + \frac{K_n}{6}|u|^3$$

with $K_n = 4/3, 6, 15$ in the three cases. Since $\sum_j \log x_j = 0$, the linear terms cancel. If $\delta = \max_j |\log x_j|$, then

$$S_n^m(x) \leq \left(-\frac{n}{4} + \frac{K_n \delta}{6}\right) \sum_j (\log x_j)^2.$$

The three radial hypotheses imply respectively

$$\delta \leq \log 3 < \frac{9}{8}, \quad \delta \leq \log \frac{9}{7} < \frac{1}{3}, \quad \delta \leq \log \frac{17}{15} < \frac{1}{6}.$$

The displayed coefficient is therefore strictly negative in every case, unless every logarithm is zero. $\square$

Theorem 8.7 (Complete minimum-dimension failure staircase). *For $n \geq 1$, let*

$$d_n = \min\{m \geq 1 : \mathcal{I}_n^m \not\subseteq \mathcal{I}_{n+1}^m\}.$$

Then

$$d_n = \begin{cases} 12, & n = 1, \\ 7, & n = 2, \\ 6, & n = 3, \\ 5, & 4 \leq n \leq 6, \\ 4, & n \geq 7. \end{cases}$$

Thus every integer one-step inclusion is classified by the single staircase above.

Proof. The upper bounds are the exact rational witnesses in Section 12; appending coordinates equal to 1 propagates every failure to larger dimensions. The cases $4 \leq n \leq 6$ are sharp because Corollary 8.5 proves the four-variable inclusions, while the displayed five-variable witnesses fail. For $n \geq 7$, Theorem 8.4 gives failure already in dimension four, and dimensions at most three are universally nested.

It remains to establish the lower bounds for $n = 1, 2, 3$. These are finite computer-assisted theorems. On a strict sign sector write the coordinates above 1 as $t_i > 1$ and the reciprocals of the coordinates below 1 as $a_j > 1$. The product constraint becomes

$$\prod_i t_i = \prod_j a_j.$$

Use the compact radial coordinate

$$y = \frac{t-1}{t+1} \in [0, 1),$$

and define the positive contribution $p_n(y) = \phi_n(t)$ and reciprocal cost $q_n(y) = t\phi_n(t)$. For the required exponents,

$$\begin{aligned} p_1(y) &= \frac{y(1-y)}{1+y^2}, & q_1(y) &= \frac{y(1+y)}{1+y^2}, \\ p_2(y) &= \frac{2y(1-y)}{1+3y^2}, & q_2(y) &= \frac{2y(1+y)}{1+3y^2}, \\ p_3(y) &= \frac{y(1-y)(y^2+3)}{y^4+6y^2+1}, & q_3(y) &= \frac{y(1+y)(y^2+3)}{y^4+6y^2+1}, \\ p_4(y) &= \frac{4y(1-y)(1+y^2)}{1+10y^2+5y^4}, & q_4(y) &= \frac{4y(1+y)(1+y^2)}{1+10y^2+5y^4}. \end{aligned}$$

Lemma 8.8 (Tensor-product rational Bernstein enclosure). *Let*

$$K = \prod_{j=1}^k [a_j, b_j]$$

be a rectangular box, and let $\{B_\alpha\}_{0 \leq \alpha \leq d}$ be the tensor-product Bernstein basis of a fixed multidegree $d = (d_1, \dots, d_k)$ on K . Suppose that, after degree elevation to this common multidegree,

$$p(x) = \sum_{\alpha} p_{\alpha} B_{\alpha}(x), \quad q(x) = \sum_{\alpha} q_{\alpha} B_{\alpha}(x), \quad q_{\alpha} > 0.$$

Then $q(x) > 0$ throughout K , and

$$\frac{p(x)}{q(x)} = \sum_{\alpha} \frac{p_{\alpha}}{q_{\alpha}} \frac{q_{\alpha} B_{\alpha}(x)}{\sum_{\beta} q_{\beta} B_{\beta}(x)}.$$

The weights on the right are nonnegative and sum to one. Consequently

$$\min_{\alpha} \frac{p_{\alpha}}{q_{\alpha}} \leq \frac{p(x)}{q(x)} \leq \max_{\alpha} \frac{p_{\alpha}}{q_{\alpha}}.$$

Proof. Every tensor-product Bernstein basis function is nonnegative on K and the basis partitions unity. Thus $q(x) > 0$, and the displayed identity follows by multiplying each ratio p_{α}/q_{α} by $q_{\alpha} B_{\alpha}(x)/q(x)$ and summing. The resulting weights form a convex combination. $\square$

Each sign sector is partitioned into dyadic y -boxes, with coordinates grouped by permutation symmetry. On every rational box, numerator and denominator are converted exactly to tensor-product Bernstein form and degree-elevated to a common multidegree. Positivity of the denominator coefficients and Lemma 8.8 give the rational range enclosure used below. A box is certified whenever one of the following exact conditions holds:

- (a) its positive- and negative-radius product intervals are disjoint;
- (b) the upper enclosure for S_n is nonpositive;
- (c) the lower enclosure for S_{n+1} is positive;
- (d) the upper enclosure for $cS_n - S_{n+1}$ is nonpositive for one of the fixed rational values

n	allowed c
1	$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$
2	$\frac{5}{3}$
3	$\frac{3}{2}$

(the verifier tests this finite list exactly and records success as soon as one value applies);

- (e) it lies in the local region of Lemma 8.6.

Criterion (d) is valid because $S_n > 0$ then implies $S_{n+1} \geq cS_n > 0$. The verifier subdivides only unresolved boxes, splitting a symmetric count group among its two dyadic children; hence every permutation is covered without enumerating labelled coordinates.

For $n = 1$, it is enough to certify the maximal candidate safe dimension $m = 11$. Indeed, any failure in a smaller dimension would lift to dimension 11 by appending coordinates equal to 1. Although the certificates are indexed by strict sign counts, every dyadic radial chart is certified on its closure $0 \leq y \leq 1$. A coordinate with $y = 0$ is exactly a unit coordinate, so the archived maximal-dimensional exhaustion already contains all lower-dimensional coordinate traces arising from appended 1's; there is no circular appeal to an unverified smaller-dimensional

case. Exact exhaustion of all eight nontrivial sign-count charts $r + s = 11$, $2 \leq r, s \leq 9$, proves $\mathcal{I}_1^{11} \subseteq \mathcal{I}_2^{11}$. The one-above sectors are impossible by Theorem 3.4, and the one-below sectors persist analytically by Theorem 3.5. Hence $d_1 \geq 12$.

The same maximal-dimension reduction shortens the remaining lower bounds. For $n = 2$ it suffices to certify the three strict nontrivial sectors in dimension $m = 6$; every smaller failure would lift to $m = 6$. For $n = 3$ it suffices to certify the two strict nontrivial sectors $2 + 3$ and $3 + 2$ in dimension $m = 5$. The corresponding one-below sectors are analytic, and the lower-dimensional coordinate strata are inherited by appending 1's. Thus $d_2 \geq 7$ and $d_3 \geq 6$.

All range arithmetic, dyadic endpoints, product tests, and pruning inequalities are rational. The released verifier performs a deterministic exhaustive subdivision directly from the closed radial formulas. It begins with every permutation-symmetric depth-two radial box, applies criteria (a)–(e) exactly, and subdivides only an unresolved box. It returns failure rather than accepting a box if the prescribed depth or node budget is exhausted. Consequently, successful termination proves that the listed terminal criteria cover every initial box; no floating-point decision or unrecorded numerical tolerance enters. Independent top-level boxes may be distributed among worker processes, but the search within each box and every sign decision are deterministic. The exact rational counterexamples furnish the opposite inequalities. The reproducible node counts are shown in Table 2.

Table 2. Deterministic exact staircase exhaustions reproduced by the released verifier.

n	sector	exact nodes	top-level boxes
1	2 + 9	3,181	2,200
1	3 + 8	7,156	3,300
1	4 + 7	14,749	4,200
1	5 + 6	24,567	4,704
1	6 + 5	34,772	4,704
1	7 + 4	78,931	4,200
1	8 + 3	631,726	3,300
1	9 + 2	565,317	2,200
2	2 + 4	54,249	350
2	3 + 3	129,021	400
2	4 + 2	56,746	350
3	2 + 3	175,022	200
3	3 + 2	210,259	200

This proves the staircase. □

Corollary 8.9 (Asymptotic size of the four-variable sign flip). *For the family X_n in the proof of Theorem 8.4, with $N = n + 1$ and $L = \log(4/3)$,*

$$NS_n^4(X_n) \longrightarrow \frac{3L}{7}, \quad NS_{n+1}^4(X_n) \longrightarrow -\frac{3L}{49}.$$

Thus the persistent failure of one-step nesting has a nonzero first-order $1/n$ profile.

Proof. The corrections from the fixed coordinates $16/(49r)$ and $7/4$ are exponentially small. Since $r = e^{-L/N}$,

$$N \left(\frac{1}{r} - 1 \right) \rightarrow L, \quad N(1 - r) \rightarrow L.$$

The baseline identities used in Theorem 8.4 are

$$-1 + \phi_n(r) + \frac{8}{7} = \frac{3}{7} \left(\frac{1}{r} - 1 \right)$$

and

$$-1 + \phi_{n+1}(r) + \frac{8}{7} = -\frac{3(1 - r)}{7(4 + 3r)}.$$

Multiplying by N gives the stated limits. □

Example 8.10 (Re-entrant exponent behaviour). Set

$$Y = \left(\frac{4}{7}, \frac{9}{10}, \frac{5}{4}, \frac{3}{2}, \frac{28}{27} \right).$$

Then $\prod_j Y_j = 1$. Exact rational evaluation gives

$$S_{11}^5(Y) < 0 < S_{12}^5(Y),$$

$$S_{18}^5(Y) > 0 > S_{19}^5(Y),$$

$$S_{23}^5(Y) < 0 < S_{24}^5(Y).$$

Thus the membership history contains

$$\text{out} \longrightarrow \text{in} \longrightarrow \text{out} \longrightarrow \text{in}.$$

Moreover

$$L(Y) = \frac{181}{420} > 0,$$

so the final re-entry persists by Theorem 4.2.

The preceding example is not an isolated finite-dimensional accident: there is no dimension-free bound on re-entry complexity.

Theorem 8.11 (Arbitrarily many exponent re-entries). *For every integer $N \geq 1$, there are $m = 4(N + 1)$ and a point $x \in \mathcal{H}_m$ for which the membership sequence*

$$\mathbf{1}_{\{S_n^m(x) > 0\}}, \quad n = 1, 2, \dots,$$

changes value at least N times. The construction uses at most $3(N + 1)$ distinct reciprocal radii.

Proof. Introduce the positive pulse

$$H(t) = -\tanh \frac{t}{2} + 2 \tanh t - \tanh \frac{3t}{2}, \quad t > 0.$$

If $u = \tanh(t/2) \in (0, 1)$, the double- and triple-angle formulas give

$$H(t) = \frac{4u^3(1 - u^2)}{(1 + u^2)(1 + 3u^2)} > 0.$$

Moreover $H(t) \rightarrow 0$ both as $t \downarrow 0$ and as $t \rightarrow \infty$.

Fix an integer $R > 3$, to be chosen large, and set

$$F_R(t) = \sum_{j=0}^N (-1)^j H(R^j t).$$

At the sample points $t_j = R^{-j}$,

$$F_R(t_j) = \sum_{\ell=0}^N (-1)^\ell H(R^{\ell-j}).$$

For fixed N , letting $R \rightarrow \infty$ makes every term with $\ell \neq j$ tend to zero, while the remaining term is $(-1)^j H(1)$. Hence one can choose an integer $R > 3$ such that the $N + 1$ values $F_R(t_j)$ have strictly alternating signs.

Expand every pulse into three radial frequencies. For $0 \leq j \leq N$ and $k = 1, 2, 3$, put

$$w_{j,k} = kR^j, \quad (c_{j,1}, c_{j,2}, c_{j,3}) = (-1)^j(-1, 2, -1).$$

Each block satisfies the exact product-moment identity

$$\sum_{k=1}^3 c_{j,k} w_{j,k} = (-1)^j R^j (-1 + 2 - 1) = 0.$$

For $\varepsilon > 0$, construct $x(\varepsilon)$ by placing $c_{j,k}$ copies of $e^{w_{j,k}\varepsilon}$ when $c_{j,k} > 0$ and $-c_{j,k}$ copies of $e^{-w_{j,k}\varepsilon}$ when $c_{j,k} < 0$. There are four coordinates per block, and the moment identities give $\prod_i x_i(\varepsilon) = 1$ exactly.

If $n_\varepsilon(t) = \lfloor t/\varepsilon \rfloor$, then for every fixed $w, t > 0$,

$$\phi_{n_\varepsilon(t)}(e^{w\varepsilon}) \longrightarrow \tanh \frac{wt}{2}, \quad \phi_{n_\varepsilon(t)}(e^{-w\varepsilon}) \longrightarrow -\tanh \frac{wt}{2}.$$

Consequently

$$S_{n_\varepsilon(t)}^m(x(\varepsilon)) \longrightarrow \sum_{j,k} c_{j,k} \tanh \frac{w_{j,k}t}{2} = F_R(t).$$

Choose ε sufficiently small that these signs agree at all t_j and that the integers $n_\varepsilon(t_j)$ are distinct. Read the sample points in increasing order. Their membership indicators alternate, so the intervening integer-exponent sequence changes value at least N times. Since $R > 3$, the three frequencies in different blocks are distinct, giving at most $3(N+1)$ reciprocal radii. $\square$

The preceding separated-scale argument spends three fresh reciprocal radii on every pulse. The same pulse admits a sharper explicit packing because adjacent three-frequency blocks can overlap.

Theorem 8.12 (Telescoping-pulse complexity bound). *For every integer $N \geq 1$, there is a point in dimension $2N+4$ whose membership sequence changes value at least N times and which uses at most*

$$\boxed{N+3}$$

distinct reciprocal radii.

Proof. Retain the positive pulse H from Theorem 8.11, put $c = \log 3$, and define

$$F_N(t) = \sum_{j=0}^N (-1)^j H(3^j t).$$

We prove that the values at $t_j = c3^{-j}$, $0 \leq j \leq N$, have signs $(-1)^j$. The exact value at the central scale is

$$H(c) = \frac{6}{35}.$$

Two elementary global bounds control every other scale. From the rational formula in Theorem 8.11, with $u = \tanh(t/2) < t/2$, one has

$$0 < H(t) < \frac{t^3}{2}.$$

Writing $y = e^{-t}$ gives also

$$H(t) = \frac{2y(1-y)^3}{(1+y)(1+y^2)(1-y+y^2)} < 2e^{-t}.$$

Consequently the total contribution from all smaller scales is bounded by

$$\sum_{h \geq 1} H(c3^{-h}) < \frac{c^3}{2} \sum_{h \geq 1} 3^{-3h} = \frac{c^3}{52} < \frac{1331}{52000},$$

where $c = \log 3 < 11/10$. For the larger scales use $3^h \geq 1 + 2h$ to obtain

$$\sum_{h \geq 1} H(c3^h) < 2 \sum_{h \geq 1} e^{-c3^h} \leq 2 \sum_{h \geq 1} 3^{-1-2h} = \frac{1}{12}.$$

Thus

$$H(c) - \sum_{h \neq 0} H(c3^h) > \frac{6}{35} - \frac{1331}{52000} - \frac{1}{12} = \frac{68249}{1092000} > 0.$$

At $t = t_j$, the j th summand therefore strictly dominates the sum of the absolute values of all other summands, proving

$$(-1)^j F_N(t_j) > 0.$$

Expand the j th pulse as in Theorem 8.11. Its frequencies are $\{3^j, 2 \cdot 3^j, 3^{j+1}\}$, with signed multiplicities $(-1)^j(-1, 2, -1)$. Consecutive endpoint coefficients cancel: for $1 \leq j \leq N$, the coefficient of 3^j from block $j-1$ is $(-1)^j$, while that from block j is $(-1)^{j+1}$. After collecting equal frequencies, the only nonzero signed multiplicities are therefore

$$c_1 = -1, \quad c_{3N+1} = (-1)^{N+1}, \quad c_{2 \cdot 3^j} = 2(-1)^j \quad (0 \leq j \leq N).$$

There are $N+3$ occupied frequencies and the sum of the absolute multiplicities is

$$1 + 1 + 2(N+1) = 2N + 4.$$

Moreover their signed first moment is zero because it is the sum of the block identities

$$(-1)^j 3^j (-1 + 4 - 3) = 0.$$

The discrete lifting from Theorem 8.11, now applied directly to these collected coefficients, therefore gives a point of $\mathcal{H}_{2N+4}$ with the prescribed alternating samples and at least N membership changes. $\square$

9 Universal microscopic geometry

Lemma 9.1 (Uniform n^{-2} blow-up). *Fix m and let $v = (v_1, \dots, v_m)$ range over a bounded subset of the hyperplane $\sum_j v_j = 0$. Then uniformly on bounded sets,*

$$n^3 S_n^m \left(e^{v_1/n^2}, \dots, e^{v_m/n^2} \right) = -\frac{1}{24} Q_m(v) + O \left(\frac{\sum_j v_j^2}{n} \right), \quad (27)$$

where

$$Q_m(v) = \sum_{j=1}^m (v_j^3 + 6v_j^2).$$

Proof. Taylor expansion of $g_n(u) = \phi_n(e^u)$ at 0 gives

$$g_n(u) = \frac{n}{2}u - \frac{n}{4}u^2 - \frac{n^2(n+3)}{24}u^3 + \frac{n(n^2+3n+1)}{48}u^4 + O(n^5|u|^5)$$

uniformly when $|nu|$ is small. Put $u = v/n^2$. The linear terms cancel because $\sum_j v_j = 0$; collecting the quadratic and cubic terms gives (27), with the stated relative remainder on bounded v . $\square$

The cubic profile yields a sharp extremal problem.

Lemma 9.2 (Sharp sign-count cubic moment inequality). *Let $v \in \mathbb{R}^m$ satisfy $\sum_j v_j = 0$, and suppose exactly $r \geq 1$ coordinates of v are positive. Then*

$$-\sum_j v_j^3 \leq \frac{r-1}{r} \|v\|_\infty \sum_j v_j^2. \quad (28)$$

The constant is sharp. After normalising $\|v\|_\infty = 1$, equality in the closure of the sign sector occurs, up to permutation, at

$$\left(-1, \underbrace{\frac{1}{r}, \dots, \frac{1}{r}}_{r \text{ positive entries}}, 0, \dots, 0 \right).$$

If the sector has more than one negative coordinate, the same constant is approached by replacing the zero entries by arbitrarily small negative entries and adjusting the r positive entries equally.

Proof. By homogeneity assume $\|v\|_\infty = 1$. Write the positive coordinates as $p_1, \dots, p_r$ and the absolute values of the negative coordinates as $q_1, \dots, q_s$. Their common total is

$$T := \sum_{i=1}^r p_i = \sum_{j=1}^s q_j.$$

Because some coordinate has absolute value 1, one has $T \geq 1$. Put $c = (r-1)/r$. Power-mean/Cauchy estimates give

$$\sum_i p_i^2 \geq \frac{T^2}{r}, \quad \sum_i p_i^3 \geq \frac{T^3}{r^2}.$$

Also $0 \leq q_j \leq 1$, so $q_j^3 \leq q_j^2$ and $\sum_j q_j^2 \leq T$. Therefore

$$\begin{aligned} \sum_j v_j^3 + c \sum_j v_j^2 &= \sum_i (p_i^3 + c p_i^2) + \sum_j (c q_j^2 - q_j^3) \\ &\geq \frac{T^3}{r^2} + \frac{(r-1)T^2}{r^2} - \frac{T}{r} \\ &= \frac{T(T-1)(T+r)}{r^2} \geq 0. \end{aligned}$$

This is (28). Equality requires $T = 1$, equality in the positive power-mean estimates, and $q_j \in \{0, 1\}$ with total 1, yielding the displayed extremal vector. The approximation statement is immediate. $\square$

Corollary 9.3 (Sharp finite-sample cubic moment inequality). *For every $v \in \mathbb{R}^m$ with $\sum_j v_j = 0$,*

$$-\sum_j v_j^3 \leq \frac{m-2}{m-1} \|v\|_\infty \sum_j v_j^2. \quad (29)$$

The constant is sharp, with equality apart from scaling and permutation at

$$\left(-1, \frac{1}{m-1}, \dots, \frac{1}{m-1} \right).$$

Proof. A nonzero zero-sum vector has at most $m - 1$ positive coordinates. The coefficient $(r - 1)/r$ in Lemma 9.2 increases with r , so (29) follows by taking $r \leq m - 1$. Equality forces $r = m - 1$ and the equality case of Lemma 9.2. $\square$

Theorem 9.4 (Sharp sign-sector distance from the identity). *Fix $r \geq 2$, $s \geq 1$, and $m \geq r + s$. Let $d_n^{(r,s)}$ be the infimum of $\max_j |\log x_j|$ over violating points having exactly r coordinates above 1, exactly s below 1, and the remaining coordinates equal to 1. Then this sector is nonempty for all sufficiently large n and*

$$\boxed{n^2 d_n^{(r,s)} \longrightarrow D_r := \frac{6r}{r-1}.} \quad (30)$$

In particular the local nucleation scale depends only on the number of coordinates above 1, not on s or on the ambient dimension.

Proof. For a zero-sum log-vector v in this sign sector, Lemma 9.2 gives, with $R = \|v\|_\infty$,

$$Q_m(v) = 6 \sum_j v_j^2 + \sum_j v_j^3 \geq \left(6 - \frac{r-1}{r} R\right) \sum_j v_j^2.$$

Thus $Q_m(v) > 0$ whenever $0 < R < D_r$. The uniform blow-up formula Lemma 9.1 excludes violations with $n^2 \max |\log x_j| \leq D_r - \delta$ for all sufficiently large n .

For the matching upper bound, first take the closure direction

$$w_0 = \left(-1, \underbrace{1/r, \dots, 1/r}_{r \text{ times}}, 0, \dots, 0 \right).$$

For any $R > D_r$, one has $Q_m(Rw_0) < 0$. If $s > 1$, perturb the zero negative entries to small negative numbers and adjust the r positive entries equally; for a sufficiently small perturbation the resulting zero-sum vector w has exactly the required sign counts and still satisfies $Q_m(Rw) < 0$. Then Lemma 9.1 produces a violating point in the prescribed sector for every sufficiently large n . Letting $R \downarrow D_r$ completes the proof. $\square$

Theorem 9.5 (Sharp distance from the identity). *For fixed $m \geq 3$, define*

$$d_n^m = \inf_{x \in \mathcal{I}_n^m} \max_j |\log x_j|.$$

Then

$$\boxed{n^2 d_n^m \longrightarrow C_m := \frac{6(m-1)}{m-2}.} \quad (31)$$

More precisely, call a sequence $x^{(n)} \in \mathcal{I}_n^m$, defined for all sufficiently large n , asymptotically distance-minimising if

$$\frac{\max_j |\log x_j^{(n)}|}{d_n^m} \longrightarrow 1.$$

For every such sequence, after applying permutations and passing to a subsequence,

$$n^2 (\log x_1^{(n)}, \dots, \log x_m^{(n)}) \longrightarrow \left(-C_m, \frac{C_m}{m-1}, \dots, \frac{C_m}{m-1} \right).$$

Proof. Every violation has at least two coordinates above 1 by Theorem 3.4, and at most $m - 1$. Since $D_r = 6r/(r - 1)$ decreases strictly with r , Theorem 9.4 shows that the smallest sectorwise constant is attained at $r = m - 1$, giving $C_m = D_{m-1}$. This proves the limit.

For the direction statement, take any asymptotically distance-minimising sequence and scale its logarithms by n^2 . By the definition above and the already proved limit for $n^2 d_n^m$, these vectors have sup norms converging to C_m and are therefore bounded. Pass to a convergent subsequence. Positivity together with Lemma 9.1 forces $Q_m \leq 0$ at the limit, while (29) with $R = C_m$ forces $Q_m \geq 0$. Hence equality holds in the sharp moment inequality, giving the displayed limit after a permutation. $\square$

In particular,

$$C_3 = 12, \quad C_4 = 9, \quad C_5 = 8, \quad C_m \downarrow 6.$$

The sectorwise constants sharpen this picture. In four variables, the persistent $3 + 1$ sector has nucleation constant $D_3 = 9$, whereas the transient $2 + 2$ sector has the larger constant $D_2 = 12$. Thus the violations born closest to the identity are persistent; transient pockets first appear farther out in logarithmic distance. This separation of local scales mirrors the sector escape energies in Proposition 6.5.

10 Sharp separation scale and macroscopic growth for $m = 3$

For $m = 3$, write a violating point as

$$(e^a, e^b, e^{-a-b}), \quad 0 < a \leq b.$$

Then the logarithmic distance to the union of the coordinate planes is a .

Lemma 10.1 (Quantified small- w boundary layer). *For every $K > 0$ there exist constants $C_K > 0$, $\eta_K > 0$, and N_K such that the following holds. If*

$$n \geq N_K, \quad 0 \leq A \leq K, \quad 0 < w \leq \eta_K, \quad B := nw \geq 1, \quad A \leq B/2,$$

and

$$a = \frac{A}{n^2}, \quad b = \frac{w}{n},$$

then

$$\left| S_n^3(e^a, e^b, e^{-a-b}) - \frac{w^2}{8n}(A - 4) \right| \leq C_K \frac{w^2}{n} \left(w + \frac{1}{n} + \frac{1}{B} \right). \quad (32)$$

Proof. Put $g_n(u) = \phi_n(e^u)$ and

$$F_n(a, b) = g_n(a) + g_n(b) - e^{a+b} g_n(a + b).$$

Taylor's theorem in the smaller variable a gives

$$F_n(a, b) = F_n(0, b) + a \partial_a F_n(0, b) + \frac{a^2}{2} \partial_{aa} F_n(\theta, b)$$

for some $0 \leq \theta \leq a$. Set $\varepsilon = 1/n$. The exact hyperbolic formula

$$g_n(u) = e^{-u/2} \frac{\sinh(nu/2)}{\cosh((n+1)u/2)}$$

gives, without approximation,

$$g_n(\varepsilon w) = H(\varepsilon, w) := e^{-\varepsilon w/2} \frac{\sinh(w/2)}{\cosh((1+\varepsilon)w/2)}.$$

The function H and all of its derivatives are analytic on a neighbourhood of the compact rectangle $0 \leq \varepsilon \leq 1/N_K$, $0 \leq w \leq \eta_K$: its denominator is at least 1. In particular,

$$F_n(0, \varepsilon w) = -(e^{\varepsilon w} - 1)H(\varepsilon, w), \quad \partial_a F_n(0, \varepsilon w) = \frac{n}{2} - e^{\varepsilon w} (g_n(\varepsilon w) + g'_n(\varepsilon w)).$$

Taylor's theorem applied to these two explicit analytic expressions, with the remainder bounded uniformly on that rectangle, gives

$$F_n(0, b) = -\frac{w^2}{2n} (1 + E_0), \quad |E_0| \leq C_K \left(w + \frac{1}{n} \right), \quad (33)$$

$$\partial_a F_n(0, b) = \frac{nw^2}{8} (1 + E_1), \quad |E_1| \leq C_K \left(w + \frac{1}{n} + \frac{1}{B} \right). \quad (34)$$

More explicitly, direct differentiation at $w = 0$ gives

$$\partial_a F_n(0, w/n) = -\frac{w}{2} + \frac{n+3}{8} w^2 + O_K(nw^3 + w^3).$$

Thus the $1/B$ term in (34) is precisely the relative contribution of $-w/2$ to the leading term $nw^2/8$.

For the quadratic remainder, write $\theta = X/n^2$, where $0 \leq X \leq K$, and put

$$D(\varepsilon, X, w) := \partial_{aa} F_{1/\varepsilon}(\varepsilon^2 X, \varepsilon w).$$

For $\varepsilon > 0$ define

$$R_0(\varepsilon, X) := \varepsilon D(\varepsilon, X, 0), \quad R_1(\varepsilon, X, w) := \begin{cases} \varepsilon^2 \frac{D(\varepsilon, X, w) - D(\varepsilon, X, 0)}{w}, & w > 0, \\ \varepsilon^2 \partial_w D(\varepsilon, X, 0), & w = 0. \end{cases}$$

These definitions give the exact identity

$$\partial_{aa} F_n(X/n^2, w/n) = nR_0(1/n, X) + n^2 w R_1(1/n, X, w).$$

Substitution of the displayed hyperbolic formula shows that R_0 and R_1 extend continuously to $\varepsilon = 0$ on a neighbourhood of $[0, 1/N_K] \times [0, K] \times [0, \eta_K]$: the powers of ε cancel the derivatives of $\sinh(u/(2\varepsilon))$ and $\cosh((1+\varepsilon)u/(2\varepsilon))$, while the quotient defining R_1 extends by the fundamental theorem of calculus in w . Compactness therefore supplies a uniform bound for both functions and hence

$$|\partial_{aa} F_n(\theta, b)| \leq C_K(n + n^2 w). \quad (35)$$

Substituting $a = A/n^2$ into (33)–(35), the second-order Taylor remainder is bounded by

$$C_K \frac{A^2}{n^4} (n + n^2 w) \leq C_K \frac{w^2}{n} \left(\frac{1}{B} + \frac{1}{B^2} \right),$$

where $B = nw \geq 1$. Combining the three bounds and enlarging C_K proves (32). $\square$

Theorem 10.2 (Sharp coordinate-plane separation scale). *Let*

$$\sigma_n = \inf_{x \in \mathcal{I}_n^3} \min_j |\log x_j|.$$

Then

$$\boxed{n^2 \sigma_n \longrightarrow 4.} \quad (36)$$

Proof. Let $g_n(u) = \phi_n(e^u)$. By reciprocity,

$$S_n^3(e^a, e^b, e^{-a-b}) = g_n(a) + g_n(b) - e^{a+b} g_n(a+b).$$

Upper bound. Put $a = A/n^2$, $b = w/n$ with fixed $A, w > 0$. A direct expansion, uniform for w in compact subsets of $(0, \infty)$, gives

$$nS_n^3 \longrightarrow \tanh \frac{w}{2} \left(\frac{A}{2} \tanh \frac{w}{2} - w \right). \quad (37)$$

Thus violation occurs eventually whenever

$$A > \frac{2w}{\tanh(w/2)}.$$

Since the right side decreases to 4 as $w \downarrow 0$, $\limsup n^2 \sigma_n \leq 4$.

Lower bound. Suppose for contradiction that along a sequence of violations

$$A_n := n^2 a_n \leq 4 - \delta.$$

Put $w_n = nb_n$ and pass to a subsequence.

If $w_n \rightarrow w \in (0, \infty)$, (37) and

$$\frac{2w}{\tanh(w/2)} > 4$$

contradict positivity.

If $w_n \rightarrow \infty$, note that $g_n(a) \leq na/2$. The function $h_n(u) = e^u g_n(u)$ is strictly increasing for $u > 0$: indeed, with $t = e^u > 1$,

$$h_n(u) = t\phi_n(t) = \frac{t^{n+1} - t}{t^{n+1} + 1}, \quad \frac{d}{dt}(t\phi_n(t)) = \frac{nt^{n+1} + (n+1)t^n - 1}{(t^{n+1} + 1)^2} > 0.$$

Hence

$$S_n^3 \leq g_n(a) - (e^b - 1)g_n(b).$$

Also $g_n(b) > e^{-b} \tanh(nb/2)$. If b stays away from zero, the second term is bounded below by a positive constant; if $b \rightarrow 0$, then

$$n(1 - e^{-b}) \tanh(nb/2) \sim w_n \rightarrow \infty.$$

Either way the negative term dominates $g_n(a) = O(1/n)$.

Finally suppose $w_n \rightarrow 0$, and put $B_n = n^2 b_n$. We separate the two scales that are hidden inside this regime. If (B_n) is bounded, pass to a further subsequence so that $A_n \rightarrow A$ and $B_n \rightarrow B$. The global identity-distance theorem ensures $(A, B) \neq (0, 0)$, while the uniform blow-up formula applied to

$$v = (A, B, -A - B)$$

forces $Q_3(v) \leq 0$. Since

$$Q_3(v) = 12(A^2 + AB + B^2) - 3AB(A + B),$$

writing $r = A/B \in (0, 1]$ when $A > 0$ yields

$$A \geq 4 \frac{r^2 + r + 1}{r + 1} \geq 4;$$

the case $A = 0$ gives $Q_3(v) > 0$ directly. This contradicts $A_n \leq 4 - \delta$.

It remains to consider $B_n \rightarrow \infty$. Here $a_n/b_n = A_n/B_n \rightarrow 0$ while $w_n = nb_n \rightarrow 0$. Apply Lemma 10.1 with any fixed $K > 4$; its hypotheses hold for all sufficiently large n . Since

$$w_n + \frac{1}{n} + \frac{1}{B_n} \rightarrow 0,$$

the quantified estimate gives

$$S_n^3(e^{a_n}, e^{b_n}, e^{-a_n - b_n}) = \frac{w_n^2}{8n} (A_n - 4 + o(1)).$$

Positivity therefore forces $A_n \geq 4 + o(1)$, contradicting $A_n \leq 4 - \delta$. Hence $\liminf n^2 \sigma_n \geq 4$. $\square$

We next quantify the opposite, macroscopic end of the same bounded set.

Theorem 10.3 (Outer-radius asymptotic). *For $n \geq 4$, let*

$$R_n = \sup_{x \in \mathcal{I}_n^3} \max_j x_j.$$

Then

$$R_n(1 - M_n) \longrightarrow 1. \quad (38)$$

Consequently $R_n \sim n / \log n$.

Proof. Order a violating point as $y \geq x > 1 > z$. Since $z = (xy)^{-1} \leq 1/y$ and ϕ_n is increasing on $(0, 1)$,

$$S_n^3 \leq \phi_n(x) + \phi_n(y) + \phi_n(1/y) = \phi_n(x) - (y - 1)\phi_n(y) \leq M_n - (y - 1)\phi_n(y).$$

Thus a violation requires

$$H_n(y) := (y - 1)\phi_n(y) < M_n.$$

The derivative of H_n has positive numerator

$$y^{2n} - 1 + ny^n(y - y^{-1}),$$

so H_n is strictly increasing on $(1, \infty)$. Let B_n solve $H_n(B_n) = M_n$. Then $R_n \leq B_n$. Since

$$H_n(B) = \left(1 - \frac{1}{B}\right) \frac{1 - B^{-n}}{1 + B^{-(n+1)}},$$

and $M_n \rightarrow 1$, necessarily $B_n \rightarrow \infty$. Hence

$$\frac{1 - B_n^{-n}}{1 + B_n^{-(n+1)}} = 1 + O(B_n^{-n}),$$

so the identity $H_n(B_n) = M_n$ gives

$$1 - M_n = \frac{1}{B_n} + O(B_n^{-n}).$$

Therefore $B_n(1 - M_n) \rightarrow 1$.

For the matching lower bound, fix $0 < c < 1$ and set

$$x = \alpha_n, \quad y = \frac{c}{1 - M_n}, \quad z = (xy)^{-1}.$$

As $n \rightarrow \infty$, $y \rightarrow \infty$, $z \rightarrow 0$, and

$$S_n^3 = M_n + \phi_n(y) + \phi_n(z) = (1 - M_n)(c^{-1} - 1) + o(1 - M_n) > 0$$

for all sufficiently large n . Hence $\liminf R_n(1 - M_n) \geq c$. Let $c \uparrow 1$. □

Proposition 10.4 (Asymptotic confinement margin). *Let $N = n + 1$ and $L = \log(2N)$. Then*

$$M_n = 1 - \frac{L + 1}{N} + \frac{L^2 + L + 1}{2N^2} + O\left(\frac{L^3}{N^3}\right). \quad (39)$$

Consequently

$$\Lambda_n^3 = -\frac{L + 1}{N} + O\left(\frac{L^2}{N^2}\right),$$

and the continuous critical dimension

$$m_c(n) := 2 + \frac{1}{M_n}$$

satisfies

$$m_c(n) = 3 + \frac{L+1}{N} + O\left(\frac{L^2}{N^2}\right).$$

In particular, $\Lambda_n^m \rightarrow m - 3$ for every fixed m .

Proof. Put

$$a_N = N \log \alpha_n, \quad F_N(a) = e^a - Ne^{a/N} - N + 1.$$

The critical equation (13) is exactly $F_N(a_N) = 0$. We first localize the true root instead of assuming an asymptotic ansatz. For $N \geq 4$,

$$F_N(L) = N + 1 - Ne^{L/N} < 1 - L < 0,$$

using $e^x > 1 + x$, whereas $L + 1 < N$ gives

$$F_N(L + 1) = 2eN - Ne^{(L+1)/N} - N + 1 > N(e - 1) + 1 > 0.$$

Hence

$$L < a_N < L + 1. \tag{40}$$

Moreover, throughout this interval,

$$F'_N(a) = e^a - e^{a/N} \geq 2N - e \geq N$$

for $N \geq 4$.

Now define the explicit approximation

$$\tilde{a}_N = L + \frac{L-1}{2N} + \frac{L^2 + 4L - 3}{8N^2}.$$

For sufficiently large N , (40) also contains $\tilde{a}_N$. Taylor expansion with remainder, using $e^L = 2N$, gives

$$\begin{aligned} e^{\tilde{a}_N} &= 2N + (L-1) + \frac{L^2 + 4L - 3 + (L-1)^2}{4N} + O\left(\frac{L^3}{N^2}\right), \\ Ne^{\tilde{a}_N/N} + N - 1 &= 2N + (L-1) + \frac{(L-1)/2 + L^2/2}{N} + O\left(\frac{L^3}{N^2}\right). \end{aligned}$$

The displayed N^{-1} coefficients are equal, so

$$F_N(\tilde{a}_N) = O\left(\frac{L^3}{N^2}\right).$$

The mean-value theorem and the lower bound $F'_N \geq N$ therefore yield the controlled root approximation

$$a_N = \tilde{a}_N + O\left(\frac{L^3}{N^3}\right). \tag{41}$$

Finally,

$$M_n = \left(1 - \frac{1}{N}\right) e^{-a_N/N}.$$

Substitution of (41) and one final Taylor expansion gives (39). The formulas for Λ_n^3 and $m_c(n)$ follow from $\Lambda_n^3 = -1 + M_n$ and $m_c(n) = 2 + 1/M_n$. $\square$

$n = 1$	$\emptyset$	$\emptyset$	$\emptyset$	$B+S$	U	$\emptyset$: empty
$n = 2, 3$	$\emptyset$	$B+S$	U	U	U	$B+S$: bounded, separated
$n = 4$	$B+S$	B	U	U	U	B : bounded, not separated
$n \geq 5$	$B+S$	U	U	U	U	U : unbounded, not separated
	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m \geq 7$	

Figure 2. Structural phase diagram for $m \geq 3$. The exceptional bounded/non-separated cell is $(m, n) = (4, 4)$; the unique global critical escape equality occurs at $(5, 2)$, inside the unbounded cell shown here.

Table 3. Structural phase diagram for nonempty $\mathcal{I}_n^m$.

parameters	bounded?	separated?
$m = 3, n \geq 4$	yes	yes
$m = 4, n = 2, 3$	yes	yes
$m = 4, n = 4$	yes	no
$m = 4, n \geq 5$	no	no
$m = 5, n \geq 2$	no	no
$m = 6, n = 1$	yes	yes
$m = 6, n \geq 2$	no	no
$m \geq 7$	no	no

11 Phase diagram and interpretation

The preceding theorems yield a compact phase diagram for nonempty violation sets.

Three observations summarize the geometry.

First, $m = 3$ is the last dimension of universal exponent persistence. The sign constraint leaves only a $2 + 1$ pattern, and Theorem 3.5 forces every violation to persist. At $m = 4$, a $2 + 2$ split becomes possible for the first time, creating genuinely transient violation pockets.

Second, $m = 3$ is also the asymptotically critical confinement dimension. Since $M_n \rightarrow 1$,

$$\Lambda_n^m \rightarrow m - 3.$$

Thus three-variable violation sets remain bounded only because their escape energy stays slightly negative, with margin of order $(\log n)/n$. This vanishing margin is precisely reflected in the macroscopic law $R_n \sim n/\log n$.

Third, the local and global geometries live on radically different scales. In $m = 3$,

$$\text{dist}_{\log}((1, 1, 1), \mathcal{I}_n^3) \sim \frac{12}{n^2}, \quad \text{dist}_{\log}(\mathcal{I}_n^3, \{x_j = 1\}) \sim \frac{4}{n^2},$$

while the outer radius diverges like $n/\log n$. Hence a bounded set simultaneously creeps microscopically toward forbidden planes and expands macroscopically toward infinity.

12 Exact computational certificates and reproducibility

All infinite-family classifications and asymptotic theorems above have analytic proofs. Exact computation enters only in finite certification: rational witnesses, the low-exponent four-variable theorem in Proposition 8.2, the early lower cells in the staircase theorem Theorem 8.7, and the

two-variable monotonicity certificate in the topology theorem Theorem 4.10. The verification supplement is designed around the principle that a computer-assisted proof should expose exact, auditable arithmetic rather than floating-point evidence [5, 7].

The four-variable Bernstein trust chain is deliberately split into four layers.

1. `damascus_bernstein_core.py`

This module implements only the closed symmetric formulas, the compactification described in the proof of Proposition 8.2, exact power-to-Bernstein conversion, and exact midpoint de Casteljau subdivision. All pruning-relevant coefficients are Python integers.

2. `damascus_bernstein_generate.py`

This generator searches for deterministic subdivision trees and writes three finite JSON files containing the terminal split paths and pruning labels. It is *not* invoked by either verifier.

3. `damascus_bernstein_verify.py`

This replay checker reads the archived tree files, reconstructs the exact integer root Bernstein tensors from the closed formulas, checks complete dyadic coverage, follows only the recorded splits, and verifies every advertised terminal sign condition. It also checks directly that all 105 coefficients for $n = 2$ and all 675 for $n = 3$ are strictly positive.

4. `damascus_bernstein_verify_independent.py`

This is a standalone second implementation. It imports no project Bernstein module, reconstructs the compact polynomials and root tensors directly from the closed formulas, and independently implements exact Bernstein conversion and de Casteljau replay. Its only use of NumPy is as an object-array container; all sign decisions are Python integers.

Thus replay does not ask either verifier to rediscover a tree or trust floating-point interval decisions. For $n = 4, 5, 6$ the archived trees contain respectively 33, 339, and 2591 nodes, with maximum depths 5, 7, and 9.

The staircase and topology checks use a different finite-verification strategy:

- `damascus_staircase_verify.py`

recomputes all thirteen exhaustive subdivisions from the closed rational radial formulas. It first verifies the exact Bernstein derivative bounds used by the local pruning criterion, checks every top-level box, and fails if an unresolved box reaches its depth or node budget;

- `damascus_topology_verify.py`

replays the archived 9-node topology certificate and independently checks the exact Sturm sign-variation table.

The portable runner `damascus_verify_all.py` resolves all paths relative to its own extracted directory and verifies `MANIFEST.sha256` both before and after the mathematical checks. Exact package versions are pinned in `requirements.txt`, and a frozen successful transcript records the tested Python version and every check, including the standalone second four-variable implementation. The default command is

```
python damascus_verify_all.py
```

and requires no absolute path. Before submission, it must succeed from a fresh extraction after verification of the SHA-256 manifest. A permanent DOI should be attached to the exact release when the archival record is assigned.

The audit responsibilities are deliberately separated. The exact rational/symbolic audit verifies the displayed rational failure witnesses, the $7 \rightarrow 8$ and $8 \rightarrow 9$ examples, the re-entrant example in Example 8.10, the critical factorization (19), and the algebraic Taylor coefficients used elsewhere. The continuous-exponent audit verifies the continuous multiplier identities and independently expands the fixed-direction critical-sphere cubic/quartic terms in Proposition 6.12. A separate standard-library rational audit replays the off-axis bifurcation in Proposition 4.11 and the half-integer continuous failure in Proposition 5.7. The staircase program begins by checking the exact local Bernstein derivative bounds used in Lemma 8.6 and then performs the global exhaustive subdivisions. These divisions are recorded explicitly in Online Resource 1.

For completeness, exact witnesses realizing the minimum dimensions in Theorem 8.7 are collected in Section 12. Each proves failure in the displayed dimension and every larger dimension by appending 1's; the matching lower bounds are the exact sector exhaustions in the proof of Theorem 8.7.

Table 4. Exact witnesses attaining the minimum one-step failure dimensions.

step	dimension	coordinates
$1 \rightarrow 2$	12	5 (nine times), $5/7, 5/7, 49/5^{11}$
$2 \rightarrow 3$	7	$3, 3, 3, 3, 3, 2/3, 1/162$
$3 \rightarrow 4$	6	$3, 3, 3, 3, 23/27, 1/69$
$4 \rightarrow 5$	5	$2, 2, 2, 7/47, 47/56$
$5 \rightarrow 6$	5	$2, 2, 2, 4/27, 27/32$
$6 \rightarrow 7$	5	$2, 2, 2, 6/7, 7/48$
$7 \rightarrow 8$	4	$70/41, 30/17, 12/35, 697/720$
$8 \rightarrow 9$	4	$5/3, 5/3, 3/8, 24/25$

13 Open problems

The integer boundedness, separation, and one-step nesting classifications are complete. Several finer questions remain natural.

1. **Fixed-exponent topology beyond $\mathcal{I}_4^3$.** Theorem 4.10 gives a complete topology theorem at the first three-variable violating exponent, while Proposition 4.11 shows that its fixed-product symmetrization mechanism genuinely breaks by exponent 7. Determine connectedness, contractibility, boundary singularities, and the attachment pattern of the off-axis tongues for general $\mathcal{I}_n^3$; also classify the topology of the bounded transient $2 + 2$ pockets in four variables.
2. **Continuous four-variable dynamics.** The integer threshold in Corollary 8.5 is sharp, Proposition 5.7 gives an exact noninteger failure, and Theorem 5.8 rules out every sufficiently long forward inclusion from each $\nu > \nu_c$, where Theorem 5.6 determines ν_c exactly. Determine the complete finite real-exponent parameter set on which $\mathcal{I}_\nu^4 \subseteq \mathcal{I}_\mu^4$ holds.
3. **Sharp re-entry complexity.** The exponential-polynomial argument bounds membership changes by $2^k - 2$ in terms of the number of distinct reciprocal radii, while Theorem 8.12 gives examples with N changes using at most $N+3$ radii and $2N+4$ coordinates. Determine the optimal bound in k and the least dimension required for a prescribed number of changes.

14 Conclusion

The generalized Damascus violation sets exhibit a phase transition that is not visible in the validity classification alone. A single one-variable maximum controls escape at infinity; a reciprocal radial

multiplier controls persistence in the exponent; and a universal cubic form controls microscopic nucleation near the identity. Dimension three lies simultaneously at the boundary of these mechanisms: it is the last universally nested dimension, the limiting confinement dimension, and the setting in which the eventual violation region has an exact convex triangular model together with sharp n^{-2} inner and $n/\log n$ outer scales.

Dimension four is the first genuinely transient dimension. Its one-step nesting threshold is exactly $n = 7$, with the positive direction through $n = 6$ certified by independently replayable exact Bernstein subdivision and explicit failures thereafter. The stronger minimum-dimension staircase is now complete: $d_1 = 12$, $d_2 = 7$, $d_3 = 6$, $d_4 = d_5 = d_6 = 5$, and $d_n = 4$ for $n \geq 7$. At fixed exponent, $\mathcal{I}_4^3$ has exactly three connected components and each is contractible, while an exact exponent-7 slice already develops off-axis violation tongues. Higher dimensions permit arbitrarily many re-entries as the dimension grows; the telescoping-pulse construction realizes N changes in dimension $2N + 4$ with at most $N + 3$ reciprocal radii. The radial decomposition nevertheless forces only finitely many membership changes at every fixed point, with an explicit exponential-polynomial bound. Extending the exponent continuously shows that the three-variable sets form a real-parameter increasing filtration with exact first nonempty parameter $\nu_c = 3.9826231561\dots$, gives an exact four-variable half-integer failure, rules out every sufficiently long forward inclusion from each $\nu > \nu_c$, and turns the discrete confinement table into unique critical curves. At the unique critical escape point $(5, 2)$, all fixed transverse directions on the second-order sphere are classified by an explicit cubic and a strictly negative quartic correction on its zero locus. Together these results answer the principal structural questions posed in [8] and organize the generalized family into a coherent geometry of confinement, persistence, transience, topology, and escape.

Statements and Declarations

Related manuscript.. The author’s unpublished submitted manuscript *Nested Violation Sets for the Generalized Damascus Inequality* (2026) proves the three-variable nestedness theorem that motivates the multiplier framework. The present manuscript is a distinct sequel devoted to higher-dimensional classification, continuous exponents, computer-assisted thresholds, topology, and asymptotic geometry. The companion manuscript is intentionally not included in the bibliography while unpublished. Its simultaneous consideration must be disclosed to the editor, who should be asked whether the journal prefers withdrawal/replacement by the comprehensive manuscript or explicitly disclosed companion treatment.

Funding and independence.. This research received no external funding. It was undertaken independently and was not commissioned or funded by The University of the West Indies.

Competing interests.. The author declares no competing interests.

Data and code availability.. All exact certificates, verification programs, symbolic audits, pinned dependencies, a frozen verification transcript, and SHA-256 release hashes required for the computer-assisted statements are supplied as Online Resource 1. A permanent archival DOI will be added if and when one is assigned.

Supplementary information.. **Online Resource 1:** *Exact verification supplement for Geometry, Escape, and Phase Transitions in Violation Sets for the Generalized Damascus Inequality.* ZIP archive containing the archived four-variable and topology proof trees, two independently implemented four-variable replay verifiers, the deterministic exact staircase verifier,

symbolic audits, pinned dependencies, a SHA-256 manifest, and a frozen successful verification transcript.

AI-assisted tools.. AI-assisted tools, including OpenAI ChatGPT and Codex, were used during exploratory computation, symbolic checking, code development, and editorial revision. The author reviewed the mathematical arguments and is responsible for supplying and validating the exact replayable certificates for the computer-assisted claims; responsibility for the manuscript and its claims remains with the author.

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